A constructive ∞-groupoid model of homotopy type theory

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TYPES 2025
13 June 2025

Outline

Background: what are ∞-groupoids?

Problem: interpret HoTT *effectively* in ∞-groupoids.

- Current state: no model does this!

A solution:

- intuition
- properties
- applications

What are ∞-groupoids?

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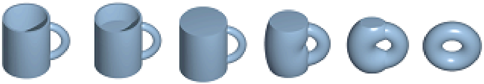
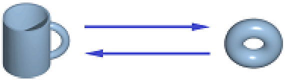
But if we have to start from **sets**?

∞-groupoids from sets

Historically: homotopy types, i.e.,

- "spaces" up to "homotopy equivalence".





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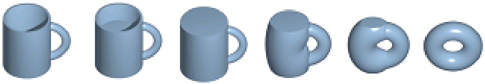
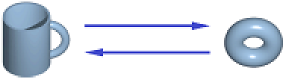
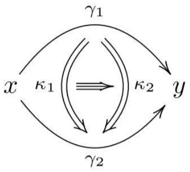




Brings point-set topology into the picture. :(

∞-groupoids from sets: higher setoids

Better: sets with a higher-dimensional equivalence relation.



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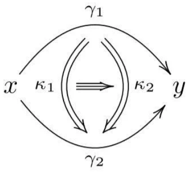

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- a set X_0 of *points*,

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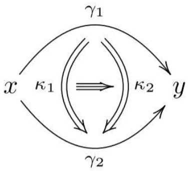

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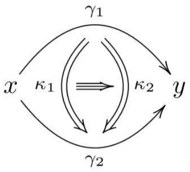

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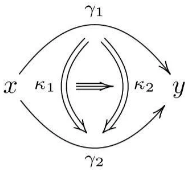

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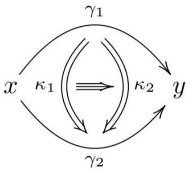

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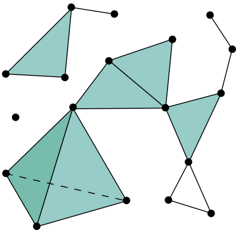
The ∞-exact completion of sets.

∞-groupoids from sets: Kan semisimplicial sets

A precise definition:
Kan semisimplicial sets up to homotopy equivalence.



Kan operation:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted, "\text{``horn filling''}"'] & \end{tikzcd}$$

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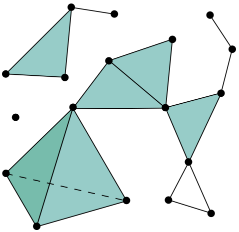

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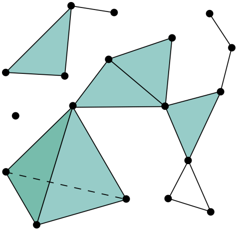

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Semisimplicial sets carry Kan weak model structure (Henry, 2020).
Quillen equivalent to simplicial sets (Henry, 2020).

∞-groupoids from sets: desired properties

A correct notion of ∞-groupoids should satisfy:

- Restricting to 0-truncated ∞-groupoids recovers setoids.
- A family over an ∞-groupoid X is contractible exactly if the fiber over each point of X is contractible over each point of X.

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Consequences:

- choice for discrete ∞-groupoids,
- presentation: every ∞-groupoid is covered by a projective one.

Types and ∞-groupoids

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But wait, hasn't this problem been solved already?

State of the art: constructive models of HoTT

By HoTT, I mean “book HoTT”: all type formers, fully split.

Ref	Setting	Model of HoTT?	Presents ∞-groupoids?
[KL21]	simplicial sets	non-constructive	✓
[BCH15]	semisimplicial sets	no	✓
[vdBF22]	effective Kan fibrations	not known	non-constructive
[GH22]	cofibrant simplicial sets	no	✓
[BCH14]	1st-generation cubical model	✓	no
[CCHM18]	2nd-generation cubical model	✓	no[a] or not known[b]
[ABC+21]	3rd-generation cubical model	✓	no
[ACC+24]	equivariant cubical model	✓	non-constructive
[CS22]	one-connection cubical sets	✓	non-constructive

[a] with reversals
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Core problem:

- Pushforward closure of fibrations requires uniformity.
- Hard to get uniformity from a higher setoid fibration.

Effective interpretation in ∞-groupoids

Starting point.

Build a cubical model ([CCHM18] or [ABC$^+$21]) in $\mathcal{P}(\square)$ for:

- fully faithful extension $j\colon \Delta \to \square$ of simplex category,
- induced interval object in $\square$ has connections,
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For example:

- presheaves over inhabited finite complete posets,
- presheaves over inhabited finite posets.

Effective interpretation in ∞-groupoids

We obtain adjunctions:

$$\begin{tikzcd} \mathcal{P}(\Delta_+)_{\mathsf{Kan}} \arrow[r, bend right, "i_*"'] \arrow[r, phantom, "\perp"] & \mathcal{P}(\Delta)_{\mathsf{Kan}} \arrow[l, bend right, "i^*"'] \arrow[r, bend right, "j_*"'] \arrow[r, phantom, "\perp"] & \mathcal{P}(\square)_{\mathsf{uniform\ Kan}} \arrow[l, bend right, "j^*"'] \end{tikzcd}$$

Write m for the composite

$$\Delta_+ \xrightarrow{i} \Delta \xrightarrow{j} \square.$$

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Consider the adjunction

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We claim:

(1) the functors m^* and m_* in (1) are (algebraic) right Quillen,

(2) the Quillen adjunction (1) is a Quillen reflection,

(3) the induced lex operation M on $\mathcal{P}(\square)$ is a lex modality (Rijke, Spitters, Shulman; 2020).

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The surprising part is (1) (see next pages).

Preservation of trivial fibrations

Key: semisimplicial sets are a "garden of uniformity".

$$\begin{tikzcd} \mathcal{P}(\square) \arrow[d, bend right, "m^*"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{P}(\Delta_+) \arrow[u, bend right, "m_*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{TF}_{\mathsf{unif}} \arrow[r] & \mathsf{TF}_{\mathsf{Kan}} \arrow[d, "m^*"] & & \mathsf{TF}_{\mathsf{unif}} \\ & \mathsf{TF}_{\mathsf{Kan}} \arrow[rr, "\text{garden of uniformity}"'] & & \mathsf{TF}_{\mathsf{unif}} \arrow[u, "m_*"'] \end{tikzcd}$$

Terminology:

- $\mathsf{TF}_{\mathsf{Kan}}$ is trivial Kan fibrations.
  These are maps with fillers for simplex boundaries.
- $\mathsf{TF}_{\mathsf{unif}}$ is uniform trivial fibrations.

Preservation of fibrations

$$\begin{tikzcd}
F_{\mathrm{unif}} \arrow[r] & F_{\mathrm{Kan}} \arrow[d, "m^*"] & & & TF_{\mathrm{unif}} \\
& F_{\mathrm{Kan}} \arrow[r] & F_{\mathrm{fill}} \arrow[r, "\text{garden of uniformity}"'] & F_{\mathrm{unif\,fill}} \arrow[r] & F_{\mathrm{unif\,comp}} \arrow[u, "m_*"']
\end{tikzcd}$$

Terminology:

- F_{Kan} is Kan fibrations.
  These are maps with fillers for horn inclusions.
- F_{fill} is prism filling fibrations in semisimplicial sets.
  These are created from trivial Kan fibrations by pullback monoidal hom with interval endpoints.
- $F_{\mathrm{unif\,fill}}$ is uniform filling fibrations.
  These are created from uniform trivial fibrations by pullback monoidal hom with interval endpoints.
- $F_{\mathrm{unif\,comp}}$ is uniform composition fibrations.

Axioms validated

Inherited from the homotopy theory of $\mathcal{P}(\Delta)_+$:

- pointwise principle,
- discrete choice,
- dependent choice,
- presentation,

Also:

- higher inductive types (similar to [CRS21])

Applications

With a homotopically correct base model, we can finally get constructive higher topos models of HoTT!

Idea: combine the model with the construction of [CRS21].
Ongoing project: constructive model of synthetic stone duality and better constructive models for synthetic algebraic geometry (joint work with Thierry Coquand and Jonas Höfer).

There are other cases, for example:

- constructive interpretation of simplicial type theory in higher categories (including the new variant of (Gratzer, Weinberger, Buchholtz; 2024),

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Also:

- higher realizability (Swan)

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