Ecumenical logic

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Motivation I – What is a proof?

Theorem 1. There exist $x, y \notin \mathbb{Q}$ such that $x^y \in \mathbb{Q}$.

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Proof. Consider $a = \sqrt{2}^{\sqrt{2}}$.
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If $a \notin \mathbb{Q}$, then take $x = a$ and $y = \sqrt{2}$. Then

$$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$$

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Intuitionistic mathematician: but $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ or $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$??? 😕

Motivation I – What is a proof?

CHAPTER **10**

THE GELFOND-SCHNEIDER THEOREM

1. Hilbert's seventh problem. In 1900 David Hilbert announced a list of twenty-three outstanding unsolved problems. The seventh problem was settled by the publication of the following result in 1934 by A. O. Gelfond, which was followed by an independent proof by Th. Schneider in 1935.

THEOREM 10.1. *If α and β are algebraic numbers with $\alpha \neq 0$, $\alpha \neq 1$, and if β is not a real rational number, then any value of α^β is transcendental.*

Remarks. The hypothesis that "β is not a real rational number" is usually stated in the form "β is irrational." Our wording is an attempt to avoid the suggestion that β must be a real number. Such a number as $\beta = 2 + 3i$, sometimes called a "complex rational number," satisfies the hypotheses of the theorem. Thus the theorem establishes the transcendence of such numbers as 2^i and $2^{\sqrt{2}}$. In general, $\alpha^\beta = \exp\{\beta \log \alpha\}$ is multiple-valued, and this is the reason for the phrase "any value of" in the statement of Theorem 10.1. One value of $i^{-2i} = \exp\{-2i \log i\}$ is e^π, and so this is transcendental according to the theorem.

Before proceeding to the proof of Theorem 10.1, we state an alternative form of the result.

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Font: *Irrational numbers*, Ivan Niven.

Motivation I – What is a proof?

Schneider theorem, and they will be given with proofs in the next section.

LEMMA 10.3. *Consider a determinant with the non-zero element ρ_j^a in the j-th row and $1 + a$-th column, with $j = 1, 2, \cdots, t$ and $a = 0, 1, \cdots, t-1$. This is called a Vandermonde determinant, and it vanishes if and only if $\rho_j = \rho_k$ for some distinct pair of subscripts j, k.*

This can be found in J. V. Uspensky, *Theory of Equations*, McGraw-Hill, p. 214. The next four lemmas are in Harry Pollard, *The Theory of Algebraic Numbers*, John Wiley, p. 53, p. 60, pp. 63–66, p. 72.

LEMMA 10.4. *Let α and β be algebraic numbers in a field K of degree h over the rationals. If the conjugates of α for K are $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_h$ and for β are $\beta = \beta_1, \beta_2, \cdots, \beta_h$, then the conjugates of $\alpha\beta$ and $\alpha + \beta$ are $\alpha_1\beta_1, \cdots, \alpha_h\beta_h$ and $\alpha_1 + \beta_1, \cdots, \alpha_h + \beta_h$.*

LEMMA 10.5. *If α is an algebraic number, then there is a positive rational integer r such that $r\alpha$ is an algebraic integer.*

LEMMA 10.6. *If K is an algebraic number field of degree h over the rationals, then there exist integers $\beta_1, \beta_2, \cdots, \beta_h$ in K such that every integer in K is expressible uniquely as a linear combination $g_1\beta_1 + \cdots + g_h\beta_h$ with rational integral coefficients. The numbers β_j are called an integral basis for K, and the discriminant of such a basis is a non-zero rational integer.*

LEMMA 10.7. *If α is an algebraic number in a field K of degree h over the rationals, then the norm $N(\alpha)$, defined as the product of α and its conjugates, satisfies the relation $N(\alpha\beta) = N(\alpha) \cdot N(\beta)$. Also $N(\alpha) = 0$ if and only if $\alpha = 0$. If α is an algebraic integer, then $N(\alpha)$ is a rational integer. If α is rational, then $N(\alpha) = \alpha^h$.*

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Motivation I – What is a proof?

Finally, from complex variable theory we need the concept of entire function, i.e., a function that is analytic in the whole complex plane, and Cauchy's residue theorem. These ideas can be found, for example, in K. Knopp's *Theory of Functions*, vol. I, Dover, p. 112ff. and p. 130.

3. Two lemmas. LEMMA 10.8. *Consider the m equations in n unknowns*

(10.1)

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = 0, \qquad k = 1, 2, \cdots, m,$$

with rational integral coefficients a_{ij}, and with $0 < m < n$. Let the positive integer A be an upper bound of the absolute values of all coefficients; thus $A \geqq |a_{ij}|$ for all i and j. Then there is a non-trivial solution $x_1, x_2, \cdots, x_n$ in rational integers of equations (10.1) *such that*

$$|x_j| < 1 + (nA)^{m/(n-m)}, \qquad j = 1, 2, \cdots, n.$$

Proof. Write y_k for $a_{k1}x_1 + \cdots + a_{kn}x_n$ so that to each point $x = (x_1, x_2, \cdots, x_n)$ there corresponds a point $y = (y_1, y_2, \cdots, y_m)$. A point such as x is said to be a *lattice point* if its coordinates x_j are rational integers. If x is a lattice point, then the corresponding point y is also a lattice point because the a_{ij} are rational integers. Let q be any positive integer. Let x range over the $(2q + 1)^n$ lattice points inside or on the n-dimensional cube defined by $|x_j| \leqq q$ for all subscripts j. Then the corresponding values of y_k satisfy

$$|y_k| = \left| \sum_{j=1}^{n} a_{kj}x_j \right| \leqq \sum_{j=1}^{n} |a_{kj}| \cdot |x_j| \leqq \sum_{j=1}^{n} Aq = nAq.$$

Thus, as x ranges over the $(2q + 1)^n$ lattice points as indicated, the corresponding lattice points y have coordinates y_k which are integers among the $2nAq + 1$

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Motivation I – What is a proof?

Sec. 3 TWO LEMMAS 139

LEMMA 10.9. *Consider the p equations in q unknowns*

(10.4)

$$\alpha_{k1}\xi_1 + \alpha_{k2}\xi_2 + \cdots + \alpha_{kq}\xi_q = 0, \qquad k = 1, 2, \cdots, p,$$

with coefficients α_{ij} which are integers in an algebraic number field K of finite degree. Assume that $0 < p < q$. Let $A \geqq 1$ be an upper bound for the absolute values of the coefficients and their conjugates for K, thus $A \geqq \| \alpha_{ij} \|$ for all i and j. Then there exists a positive constant c depending on the field K but independent of α_{ij}, p, and q, such that the equations (10.4) *have a non-trivial solution $\xi_1, \xi_2, \cdots, \xi_q$ in integers of the field K satisfying*

$$\| \xi_k \| < c + c(cqA)^{p/(q-p)}, \qquad k = 1, 2, \cdots, p.$$

Proof. Let h be the degree of K over the field of rational numbers, and let $\beta_1, \beta_2, \cdots, \beta_h$ be an integral basis for the field. If α is any integer of K, then by Lemma 10 6 we can express α uniquely as a linear combination of the integral basis,

$$\alpha = g_1\beta_1 + g_2\beta_2 + \cdots + g_h\beta_h,$$

with rational integral coefficients g_j. Denote the conjugates of α for K by $\alpha = \alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(h)}$, and similarly for the β_j. Taking conjugates in the last equation, by Lemma 10.4 we get

$$\alpha^{(i)} = g_1\beta_1^{(i)} + g_2\beta_2^{(i)} + \cdots + g_h\beta_h^{(i)}, \qquad i = 1, 2, \cdots, h.$$

The determinant $|\beta_j^{(i)}|$ is the discriminant of the basis, and it is not zero by Lemma 10.6. Hence we can solve these equations for the g_j as linear combinations of the $\alpha^{(i)}$, with coefficients dependent only on the basis. Taking absolute values throughout these solutions, we can write

(10.5) $$|g_j| < c_1 \| \alpha \|, \qquad j = 1, 2, \cdots, h,$$

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Motivation I – What is a proof?

Sec. 4 PROOF OF GELFOND-SCHNEIDER THEOREM 149

$$
\begin{aligned}
|\zeta| &< |\log \alpha|^{-p} \cdot \frac{p}{q} \cdot c_8^p p^{p(3-m)/2} \cdot \frac{2q}{p} \\
&< \{2c_8 |\log \alpha|^{-1}\}^p p^{p(3-m)/2} \\
&= c_9^p p^{p(3-m)/2}.
\end{aligned}
$$

With this estimate for $|\zeta|$, and that of Lemma 10.12 for its conjugates, we write, by (10.10),

$$|N(\zeta)| < c_9^p p^{p(3-m)2} (c^p p^p)^{h-1} = (c_9 c^{h-1})^p p^{-p} = c_0^p p^{-p},$$

where $c_0 = c_9 c^{h-1}$. This and Lemma 10.11 imply that

$$c_0^p p^{-p} > C^{-p}, \qquad Cc_0 > p,$$

for some positive constants independent of n and p. But this is a contradiction, because $p \geqq n$, and we can choose n arbitrarily large.

Notes on Chapter 10

The special case of Theorem 10.1 for any imaginary quadratic irrational β was established by A. O. Gelfond, *Compt. Rend. Acad. Sci. Paris*, **189** (1929), 1224–1226. The original papers establishing Theorem 10.1 are: A. O. Gelfond, *Doklady Akad. Nauk S.S.S.R.*, **2** (1934), 1–6; Th. Schneider, *J. reine angew. Math.*, **172** (1935), 65–69. The American Mathematical Society has provided an English translation (Translation Number 65) of an advanced expository paper by A. O. Gelfond, *The approximation of algebraic numbers by algebraic numbers and the theory of transcendental numbers, Uspehi Mat. Nauk* (*N.S.*), **4**, no. 4 (32), 19–49 (1949). There is an exposition of Gelfond's proof by E. Hille, *Amer. Math. Monthly*, **49** (1942), 654–661.

The proof of Theorem 10.1 given here is based on a simplification of Gelfond's proof by C. L. Siegel, *Transcendental Numbers*, Princeton, pp. 80–83.

Although the methods of Chapters 9 and 10 establish the transcendence of wide classes of numbers, there are many unsolved prob-

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Motivation II – What is ecumenism?

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Ecumenism: the search process for unicity, where different thoughts, ideas or points of view can harmonically co-exist.

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- What (really) are ecumenical systems?
- What are they good for?
- Why should anyone be interested in ecumenical systems?
- What is the real motivation behind the definition and development of ecumenical systems?

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Prawitz: what makes a connective classical or intuitionistic?

Logical inferentialism:

- the meaning of the logical constants can be specified by the rules that determine their correct use;
- proof-theoretical requirements on admissible logical rules: harmony and separability;
- pure logical systems: negation is not used in premises.

- IL: if what you mean by $(A \vee B)$ is $\neg(\neg A \wedge \neg B)$, then I can accept the validity of $(A \vee \neg A)$!

Logical motivation (dialogue by Luiz Carlos)

- IL: if what you mean by $(A \vee B)$ is $\neg(\neg A \wedge \neg B)$, then I can accept the validity of $(A \vee \neg A)$!
- CL: but I do not mean $\neg(\neg A \wedge \neg\neg A)$ by $(A \vee \neg A)$. One must distinguish the excluded-middle from the the principle of non-contradiction. When I say that Goldbach's conjecture is either true or false, I am not saying that it would be contradictory to assert that it is not true and that it is not the case that it is not true!

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- E.g.:

Quinne dagger

A	B	$A \downarrow B$
1	1	0
1	0	0
0	1	0
0	0	1

Sheffer stroke

A	B	$A \uparrow B$
1	1	0
1	0	1
0	1	1
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- IL: but you must realize that, at the end of the day, you just have one logical operator!!! (can you guess one?)
- CL: But this is not at all true! The fact that we can define one operator in terms of other operators does not imply that we don't have different operators!
  It is true that we can prove $\vdash (A \vee_c B) \Leftrightarrow \neg(\neg A \wedge \neg B)$ in the ecumenical system, but this does not mean that we don't have three different operators: $\neg$, $\vee_c$ and $\wedge$.

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if $x + y = 2z$ then $x \geq z$ or $y \geq z$.

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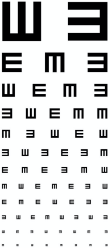

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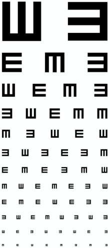

classical mathematician ☹

intuitionistic mathematician ☹

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- Prove the contrapositive $\neg q \rightarrow \neg p$ directly: assume $\neg q$, make intermediary conclusions r_1, r_2 then conclude $\neg p$. Thus, we have also established not only that $\neg q$ implies $\neg p$, but also, that it implies r_1 and r_2 etc. Thus, the proof tells us about what else must be true in worlds where q fails.

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- Prove $p \wedge \neg q \to \perp$: argue r_1, r_2, and so on, before arriving at a contradiction. The statements r_1 and r_2 are all deduced under the contradictory hypothesis, which ultimately does not hold in any mathematical situation. The proof has provided extra knowledge about a nonexistent, contradictory land.

Source: Joel David Hamkins in mathoverflow.

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- Mathematicians prefer a direct proof over a proof by contradiction.
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  - Contradiction = meet-in-the-middle strategy: explore both forwards from A and backwards from B until one gets an intersection. This is a faster strategy, with a run time which is typically the square root of the run time of the other two approaches.

Source: Terry Tao in mathoverflow.

In this talk

What makes logical connectives (including modalities) classical or intuitionistic?

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Ecumenical types! (with Delia Kesner, Mariana Milicich and Louis Riboulet)

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Ecumenical types! (with Delia Kesner, Mariana Milicich and Louis Riboulet)

(Maybe) Modalities (with Sonia Marin, Luiz Carlos Pereira and Emerson Sales)

Outline

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What is behind Ecumenism?

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For a classical logician $A \vee \neg A$ holds. For an intuitionistic logician it does not.

But why (and where) do they disagree?

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{init}}{\vdash A, \neg A}\ \neg R}{\vdash A \vee \neg A}\ \vee R \qquad \dfrac{\dfrac{\overset{?}{A \vdash \bot}}{\vdash \neg A}\ \neg R}{\vdash A \vee \neg A}\ \vee R_2$$

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- Gentzen: the problem is the disjunction!
- Maehara: the problem is the implication!
- Prawitz: the problem is both!!! ☺

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Prawitz: They are not talking about the same connective(s) (Prawitz 2015)

> *"The classical logician is not asserting what the intuitionistic logician denies: The classical logician asserts*
>
> $$A \vee_c \neg A$$
>
> *to which the intuitionist does not object; He objects to the universal validity of*
>
> $$A \vee_i \neg A,$$
>
> *which is not asserted by the classical logician."*

Outline

- Why not having a deduction system where classical and intuitionistic logic could coexist in peace?

Prawitz' idea

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- The classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation and the constant for the absurd, but they would each have their own existential quantifier, disjunction and implication, with different meanings.

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- Prawitz' main idea is that these different meanings are given by a semantical framework that can be accepted by both parties.
- The surprising aspect of Prawitz' system is its ability to share negations between the classical and the intuitionistic system, since many consider negation subject to the controversy between classical and intuitionistic logic, as implication is.

Ecumenical connectives and rules – NE

$$\dfrac{\begin{array}{c}[A, \neg B]\\ \Pi\\ \bot\end{array}}{A \to_c B}\ \to_c I$$

$$\dfrac{\begin{array}{c}[\neg A, \neg B]\\ \Pi\\ \bot\end{array}}{A \vee_c B}\ \vee_c I$$

$$\dfrac{\begin{array}{c}[\forall x.\neg A]\\ \Pi\\ \bot\end{array}}{\exists_c x.A}\ \exists_c I$$

Classical

$$\dfrac{\begin{array}{c}[A]\\ \Pi\\ \bot\end{array}}{\neg A}\ \neg I$$

$$\dfrac{A \quad B}{A \wedge B}\ \wedge I$$

$$\dfrac{A(y)}{\forall x.A}\ \forall I$$

Shared

$$\dfrac{\begin{array}{c}[A]\\ \Pi\\ B\end{array}}{A \to_i B}\ \to_i I$$

$$\dfrac{A_j}{A_1 \vee_i A_2}\ \vee_i^j I$$

$$\dfrac{A(t)}{\exists_i x.A}\ \exists_i I$$

Intuitionistic

(Prawitz 2015)

Ecumenical connectives and rules – NE

$$\dfrac{\begin{matrix}[A, \neg B] \\ \Pi \\ \bot\end{matrix}}{A \rightarrow_c B} \rightarrow_c I$$

$$\dfrac{\begin{matrix}[\neg A, \neg B] \\ \Pi \\ \bot\end{matrix}}{A \vee_c B} \vee_c I$$

$$\dfrac{\begin{matrix}[\forall x.\neg A] \\ \Pi \\ \bot\end{matrix}}{\exists_c x.A} \exists_c I$$

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Intuitionistic

(Prawitz 2015)

Provable in NE:

1. $\vdash_{\mathsf{NE}} (A \to_c \bot) \Leftrightarrow_i (A \to_i \bot) \Leftrightarrow_i (\neg A)$;
2. $\vdash_{\mathsf{NE}} (A \vee_c B) \Leftrightarrow_i \neg(\neg A \wedge \neg B)$;
3. $\vdash_{\mathsf{NE}} (A \to_c B) \Leftrightarrow_i \neg(A \wedge \neg B)$;
4. $\vdash_{\mathsf{NE}} (\exists_c x.A) \Leftrightarrow_i \neg(\forall x.\neg A)$.

Provability. . .

Provable in NE:

1. $\vdash_{\mathrm{NE}} (A \to_c \bot) \Leftrightarrow_i (A \to_i \bot) \Leftrightarrow_i (\neg A)$;
2. $\vdash_{\mathrm{NE}} (A \vee_c B) \Leftrightarrow_i \neg(\neg A \wedge \neg B)$;
3. $\vdash_{\mathrm{NE}} (A \to_c B) \Leftrightarrow_i \neg(A \wedge \neg B)$;
4. $\vdash_{\mathrm{NE}} (\exists_c x.A) \Leftrightarrow_i \neg(\forall x.\neg A)$.

However:

5. $\vdash_{\mathrm{NE}} (A \to_i B) \to_i (A \to_c B)$ but $\nvdash_{\mathrm{NE}} (A \to_c B) \to_i (A \to_i B)$ in general;
6. $\vdash_{\mathrm{NE}} A \vee_c \neg A$ but $\nvdash_{\mathrm{NE}} A \vee_i \neg A$ in general;
7. $\vdash_{\mathrm{NE}} (\neg\neg A) \to_c A$ but $\nvdash_{\mathrm{NE}} (\neg\neg A) \to_i A$ in general;
8. $\vdash_{\mathrm{NE}} (A \wedge (A \to_i B)) \to_i B$ but $\nvdash_{\mathrm{NE}} (A \wedge (A \to_c B)) \to_i B$ in general;
9. $\vdash_{\mathrm{NE}} \forall x.A \to_i \neg\exists_c x.\neg A$ but $\nvdash_{\mathrm{NE}} \neg\exists_c x.\neg A \to_i \forall x.A$ in general.

Theorem
$\Gamma \vdash A$ is provable in NE iff $\vdash_{\mathsf{NE}} \bigwedge \Gamma \rightarrow_i A$.

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Theorem
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- The Ecumenical entailment is intuitionistic!
- That is, even though some formulas carry with them the notion of classical truth, the logical consequence is intrinsically intuitionistic.
- As it should be, since the ecumenical system embeds the classical behavior into intuitionistic logic. ☺
- But if A is classical, the entailment can be read classically.
- And this justifies the ecumenical view of entailments in Prawitz's original proposal.

Peirce's law

Prove $((A \to B) \to A) \to A$ in classical logic.

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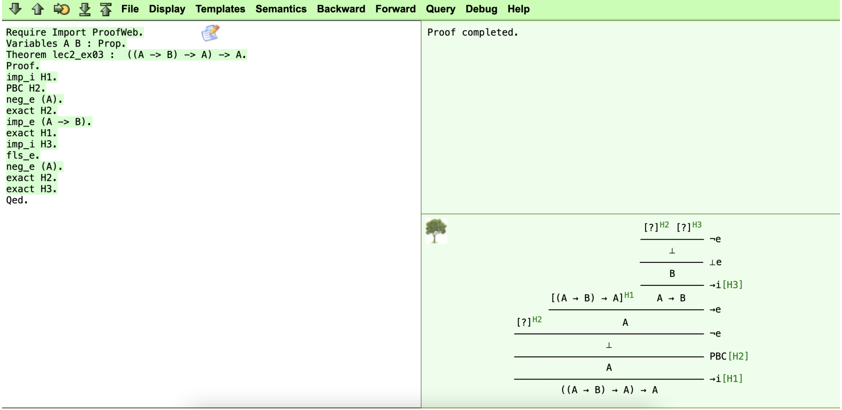

Peirce's law

Prove $((A \rightarrow_i B) \rightarrow_i A) \rightarrow_c A$ in NE.

Rules:

$$\frac{A \rightarrow_i B \quad A}{B} \rightarrow_i E \qquad \frac{\begin{array}{c}[A]\\ \Pi \\ B\end{array}}{A \rightarrow_i B} \rightarrow_i I \qquad \frac{\begin{array}{c}[A, \neg B]\\ \Pi \\ \bot\end{array}}{A \rightarrow_c B} \rightarrow_c I$$

$$\frac{A \quad \neg A}{\bot} \neg E \qquad \frac{\bot}{A} \bot E$$

Prove $((A \rightarrow_i B) \rightarrow_i A) \rightarrow_c A$ in NE.

Answer:

$$
\begin{prooftree}
\AxiomC{$[\neg A]^1$}
\AxiomC{$[A]^2$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{B}
\LeftLabel{2}
\RightLabel{$\rightarrow I$}
\UnaryInfC{$A \rightarrow_i B$}
\AxiomC{$[(A \rightarrow_i B) \rightarrow_i A]^1$}
\RightLabel{$\rightarrow_i E$}
\BinaryInfC{A}
\AxiomC{$[\neg A]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\rightarrow_c I$}
\UnaryInfC{$((A \rightarrow_i B) \rightarrow_i A) \rightarrow_c A$}
\end{prooftree}
$$

Peirce's law

Prove $((A \rightarrow_i B) \rightarrow_i A) \rightarrow_c A$ in NE.

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$$
\begin{prooftree}
\AxiomC{$[\neg A]^1$}
\AxiomC{$[A]^2$}
\RightLabel{$\neg E$}
\BinaryInfC{$\perp$}
\RightLabel{$\perp E$}
\UnaryInfC{B}
\LeftLabel{2}
\RightLabel{$\rightarrow I$}
\UnaryInfC{$A \rightarrow_i B$}
\AxiomC{$[(A \rightarrow_i B) \rightarrow_i A]^1$}
\RightLabel{$\rightarrow_i E$}
\BinaryInfC{A}
\AxiomC{$[\neg A]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\perp$}
\LeftLabel{1}
\RightLabel{$\rightarrow_c I$}
\UnaryInfC{$((A \rightarrow_i B) \rightarrow_i A) \rightarrow_c A$}
\end{prooftree}
$$

Note the occurrence of negation!!

Peirce's law

Prove $((A \to_i B) \to_i A) \to_c A$ in NE.

Answer:

$$
\begin{prooftree}
\AxiomC{$[\neg A]^1$}
\AxiomC{$[A]^2$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{B}
\LeftLabel{2}
\RightLabel{$\to I$}
\UnaryInfC{$A \to_i B$}
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\RightLabel{$\to_i E$}
\BinaryInfC{A}
\AxiomC{$[\neg A]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to_c I$}
\UnaryInfC{$((A \to_i B) \to_i A) \to_c A$}
\end{prooftree}
$$

Note the occurrence of negation!! What is negation doing there??

Outline

Negation messing up again...

NE is not pure: the definition of classical connectives depend on other connectives.

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For example:

$$\dfrac{\begin{array}{c}[\forall x.\neg A] \\ \Pi \\ \bot\end{array}}{\exists_c x.A}\ \exists_c I$$

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One way of purifying systems: polarities.

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$$\Delta;\Sigma$$

where Σ has at most one formula.

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For example:

$$\dfrac{\Delta, \exists_c x.A; A(t)}{\Delta; \exists_c x.A}\ \exists_c I$$

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$$\Delta;\Sigma$$

where Σ has at most one formula.

For example:

$$\dfrac{\Delta, \exists_c x.A; A(t)}{\Delta; \exists_c x.A}\ \exists_c I$$

Finally, for Prawitz: $p_c \equiv \neg\neg p_i$ – and this is unfortunate!

$$\dfrac{\begin{array}{c}[\cdot; A] \quad \Gamma \\ \Pi \\ \Delta, B; \cdot\end{array}}{\Delta; A \rightarrow_c B} \rightarrow_c I$$

$$\dfrac{\Delta, A, B; \cdot}{\Delta; A \vee_c B} \vee_c I$$

$$\dfrac{\Delta, \exists_c x.A; A(t)}{\Delta; \exists_c x.A} \exists_c I$$

Classical

$$\dfrac{\begin{array}{c}[\cdot; A] \quad \Gamma \\ \Pi \\ \Delta; \cdot\end{array}}{\Delta; \neg A} \neg I$$

$$\dfrac{\Delta_1; A \quad \Delta_2; B}{\Delta_1, \Delta_2; A \wedge B} \wedge I$$

$$\dfrac{\Delta; A(y)}{\Delta; \forall x.A} \forall I$$

Shared

$$\dfrac{\begin{array}{c}[\cdot; A] \quad \Gamma \\ \Pi \\ \Delta; B\end{array}}{\Delta; A \rightarrow_i B} \rightarrow_i I$$

$$\dfrac{\Delta; A_j}{\Delta; A_1 \vee_i A_2} \vee_i^j I$$

$$\dfrac{\Delta; A(t)}{\Delta; \exists_i x.A} \exists_i I$$

Intuitionistic

(Pereira & Pimentel 2022)

Ecumenical rules with *stoup* – $\mathrm{NE_s}$

$$\dfrac{\begin{matrix}[\cdot; A] & \Gamma \\ \multicolumn{2}{c}{\Pi} \\ \multicolumn{2}{c}{\Delta, B; \cdot}\end{matrix}}{\Delta; A \to_c B} \to_c I$$

$$\dfrac{\Delta, A, B; \cdot}{\Delta; A \vee_c B} \vee_c I$$

$$\dfrac{\Delta, \exists_c x.A; A(t)}{\Delta; \exists_c x.A} \exists_c I$$

Classical

$$\dfrac{\begin{matrix}[\cdot; A] & \Gamma \\ \multicolumn{2}{c}{\Pi} \\ \multicolumn{2}{c}{\Delta; \cdot}\end{matrix}}{\Delta; \neg A} \neg I$$

$$\dfrac{\Delta_1; A \quad \Delta_2; B}{\Delta_1, \Delta_2; A \wedge B} \wedge I$$

$$\dfrac{\Delta; A(y)}{\Delta; \forall x.A} \forall I$$

Shared

$$\dfrac{\begin{matrix}[\cdot; A] & \Gamma \\ \multicolumn{2}{c}{\Pi} \\ \multicolumn{2}{c}{\Delta; B}\end{matrix}}{\Delta; A \to_i B} \to_i I$$

$$\dfrac{\Delta; A_j}{\Delta; A_1 \vee_i A_2} \vee_i^j I$$

$$\dfrac{\Delta; A(t)}{\Delta; \exists_i x.A} \exists_i I$$

Intuitionistic

(Pereira & Pimentel 2022)

The idea:

$$\Gamma \vdash_{\mathrm{NE_s}} \Delta; \Sigma \quad \text{iff} \quad \Gamma, \neg\Delta \vdash_{\mathrm{NE}} \Sigma$$

Revisiting Pierce

Prove $\cdot; ((A \rightarrow_c B) \rightarrow_c A) \rightarrow_c A$ in $\mathrm{NE_s}$.

Rules:

$$\frac{\Delta; A \rightarrow_i B \quad \Delta; A}{\Delta; B} \rightarrow_i E \qquad \frac{\begin{array}{c}[\cdot; A] \\ \Pi \\ \Delta; B\end{array}}{\Delta; A \rightarrow_i B} \rightarrow_i I$$

$$\frac{\Delta; A \rightarrow_c B \quad \Delta; A \quad \begin{array}{c}[\cdot; B] \\ \Pi \\ \Delta; \cdot\end{array}}{\Delta; \cdot} \rightarrow_c E \qquad \frac{\begin{array}{c}[\cdot; A] \\ \Pi \\ \Delta, B; \cdot\end{array}}{\Delta; A \rightarrow_c B} \rightarrow_c I$$

$$\frac{\Delta; A}{\Delta, A; \cdot} \text{ der} \qquad \frac{\Delta; B}{\Delta, A; B} \text{ W}$$

Prove $\cdot; ((A \to_c B) \to_c A) \to_c A$ in $\mathrm{NE_s}$.

Answer:

$$
\begin{prooftree}
\AxiomC{$[\cdot; (A \to_c B) \to_c A]^3$}
\RightLabel{W_c}
\UnaryInfC{$A; (A \to_c B) \to_c A$}
\AxiomC{$[\cdot; A]^1$}
\RightLabel{der}
\UnaryInfC{$A; \cdot$}
\RightLabel{W_c}
\UnaryInfC{$A, B; \cdot$}
\LeftLabel{1}
\RightLabel{$\to_c I$}
\UnaryInfC{$A; A \to_c B$}
\AxiomC{$[\cdot; A]^2$}
\RightLabel{der}
\UnaryInfC{$A; \cdot$}
\LeftLabel{2}
\RightLabel{$\to_c E$}
\TrinaryInfC{$A; \cdot$}
\LeftLabel{3}
\RightLabel{$\to_c I$}
\UnaryInfC{$\cdot; ((A \to_c B) \to_c A) \to_c A$}
\end{prooftree}
$$

Revisiting Pierce

Prove $\cdot; ((A \to_c B) \to_c A) \to_c A$ in $\mathrm{NE_s}$.

Answer:

$$
3\ \dfrac{2\ \dfrac{\dfrac{[\cdot; (A \to_c B) \to_c A]^3}{A; (A \to_c B) \to_c A}\, W_c \qquad 1\ \dfrac{\dfrac{\dfrac{[\cdot; A]^1}{A; \cdot}\,\text{der}}{A, B; \cdot}\, W_c}{A; A \to_c B}\, \to_c I \qquad \dfrac{[\cdot; A]^2}{A; \cdot}\,\text{der}}{A; \cdot}\, \to_c E}{\cdot; ((A \to_c B) \to_c A) \to_c A}\, \to_c I
$$

More interestingly:

$$
\vdash_{\mathrm{NE_s}} \cdot; ((A \to_j B) \to_k A) \to_c A
$$

with $j, k \in \{i, c\}$.

Prove $\cdot; ((A \to_c B) \to_c A) \to_c A$ in $\mathrm{NE_s}$.

Answer:

$$
\dfrac{\dfrac{\dfrac{[\cdot; (A \to_c B) \to_c A]^3}{A; (A \to_c B) \to_c A}\,W_c \qquad \dfrac{\dfrac{\dfrac{[\cdot; A]^1}{A; \cdot}\,\mathrm{der}}{A, B; \cdot}\,W_c}{A; A \to_c B}\,{\to_c}\,I^{1} \qquad \dfrac{[\cdot; A]^2}{A; \cdot}\,\mathrm{der}}{A; \cdot}\,{\to_c}\,E^{2}}{\cdot; ((A \to_c B) \to_c A) \to_c A}\,{\to_c}\,I^{3}
$$

More interestingly:

$$\vdash_{\mathrm{NE_s}} \cdot; ((A \to_j B) \to_k A) \to_c A$$

with $j, k \in \{i, c\}$.

Look mom, no negation!

Prove $\cdot; ((A \to_c B) \to_c A) \to_c A$ in $\mathrm{NE_s}$.

Answer:

$$
\begin{prooftree}
\AxiomC{$[\cdot; (A \to_c B) \to_c A]^3$}
\RightLabel{W_c}
\UnaryInfC{$A; (A \to_c B) \to_c A$}
\AxiomC{$[\cdot; A]^1$}
\RightLabel{der}
\UnaryInfC{$A; \cdot$}
\RightLabel{W_c}
\UnaryInfC{$A, B; \cdot$}
\LeftLabel{1}
\RightLabel{$\to_c I$}
\UnaryInfC{$A; A \to_c B$}
\AxiomC{$[\cdot; A]^2$}
\RightLabel{der}
\UnaryInfC{$A; \cdot$}
\LeftLabel{2}
\RightLabel{$\to_c E$}
\TrinaryInfC{$A; \cdot$}
\LeftLabel{3}
\RightLabel{$\to_c I$}
\UnaryInfC{$\cdot; ((A \to_c B) \to_c A) \to_c A$}
\end{prooftree}
$$

More interestingly:

$$\vdash_{\mathrm{NE_s}} \cdot; ((A \to_j B) \to_k A) \to_c A$$

with $j, k \in \{i, c\}$.
Remember:

$$
\begin{prooftree}
\AxiomC{$[\neg A]^1$}
\AxiomC{$[A]^2$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{B}
\LeftLabel{2}
\RightLabel{$\to I$}
\UnaryInfC{$A \to_i B$}
\AxiomC{$[(A \to_i B) \to_i A]^1$}
\RightLabel{$\to_i E$}
\BinaryInfC{A}
\AxiomC{$[\neg A]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to_c I$}
\UnaryInfC{$((A \to_i B) \to_i A) \to_c A$}
\end{prooftree}
$$

What we can do with that

- Normalization

What we can do with that

- Normalization
- Curry-Howard correspondence

What we can do with that

- Normalization
- Curry-Howard correspondence
- No double negation translation (Pereira & Pimentel & de Paiva 2025)

Outline

Terms:

$$
\begin{array}{rcl}
t, s, r & ::= & x \\
& | & \lambda x.\, t \\
& | & t\,(s, x.r) \\
& | & \mu(x, \alpha).\, c \\
& | & t\,[s, x.c] \\
& | & \# c
\end{array}
$$

$\lambda_\mu \mathcal{L}\mathcal{E}_p$-calculus

Terms:

$$
\begin{array}{rcl}
t, s, r & ::= & x \\
& \mid & \lambda x.\, t \\
& \mid & t\,(s, x.r) \\
& \mid & \mu(x, \alpha).\, c \\
& \mid & t\,[s, x.c] \\
& \mid & \#c
\end{array}
$$

Commands:

$$
\begin{array}{rcl}
c & ::= & [\alpha]\, t \\
& \mid & t\,[s, x.c]
\end{array}
$$

Terms:

$$
\begin{array}{rcl}
t, s, r & ::= & x \\
 & | & \lambda x.\, t \\
 & | & t\,(s, x.r) \\
 & | & \mu(x, \alpha).\, c \\
 & | & t\,[s, x.c] \\
 & | & \# c
\end{array}
$$

Commands:

$$
\begin{array}{rcl}
c & ::= & [\alpha]\, t \\
 & | & t\,[s, x.c]
\end{array}
$$

Constructors: $\lambda x.\, t$ and $\mu(x, \alpha).\, c$

Terms:

$$
\begin{array}{rcl}
t, s, r & ::= & x \\
& \mid & \lambda x.\, t \\
& \mid & t\,(s, x.r) \\
& \mid & \mu(x, \alpha).\, c \\
& \mid & t\,[s, x.c] \\
& \mid & \#c
\end{array}
$$

Commands:

$$
\begin{array}{rcl}
c & ::= & [\alpha]\, t \\
& \mid & t\,[s, x.c]
\end{array}
$$

Constructors: $\lambda x.\, t$ and $\mu(x, \alpha).\, c$

Generalized applications: $t\,(s, x.r)$ and $t\,[s, x.r]$

$\lambda_\mu \mathcal{L}\mathcal{E}_p$-calculus

Terms:

$$\begin{array}{rcl} t, s, r & ::= & x \\ & \mid & \lambda x.\, t \\ & \mid & t\,(s, x.r) \\ & \mid & \mu(x, \alpha).\, c \\ & \mid & t\,[s, x.c] \\ & \mid & \#c \end{array}$$

Commands:

$$\begin{array}{rcl} c & ::= & [\alpha]\, t \\ & \mid & t\,[s, x.c] \end{array}$$

Constructors: $\lambda x.\, t$ and $\mu(x, \alpha).\, c$

Generalized applications: $t\,(s, x.r)$ and $t\,[s, x.r]$

Activation operator: $\#c$.

Type system

Types:

$$A, B \quad ::= \quad \alpha \mid A \to_i B \mid A \to_c B$$

Typing judgments: $\Gamma \vdash O : A\,;\,\Delta$, where O is a term or a command.

$$\frac{}{\Gamma, x : A \vdash x : A\,;\,\Delta}\ \texttt{ax}$$

$$\frac{\Gamma, x : A \vdash t : B\,;\,\Delta}{\Gamma \vdash \lambda x.\, t : A \to_i B\,;\,\Delta}\ \texttt{I-}{\to_i} \qquad \frac{\Gamma \vdash t : A \to_i B\,;\,\Delta \quad \Gamma \vdash s : A\,;\,\Delta \quad \Gamma, x : B \vdash r : C\,;\,\Delta}{\Gamma \vdash t\,(s, x.r) : C\,;\,\Delta}\ \texttt{E-}{\to_i}$$

$$\frac{\Gamma, x : A \vdash c : \bot\,;\,\Delta \cup \{\alpha : B\}}{\Gamma \vdash \mu(x, \alpha).\, c : A \to_c B\,;\,\Delta}\ \texttt{I-}{\to_c} \qquad \frac{\Gamma \vdash t : A \to_c B\,;\,\Delta \quad \Gamma \vdash s : A\,;\,\Delta \quad \Gamma, x : B \vdash c : \bot\,;\,\Delta}{\Gamma \vdash t\,[s, x.c] : \bot\,;\,\Delta}\ \texttt{E-}{\to_c}$$

$$\frac{\Gamma \vdash t : A\,;\,\Delta}{\Gamma \vdash [\alpha]\,t : \bot\,;\,\Delta \cup \{\alpha : A\}}\ \texttt{der} \qquad \frac{\Gamma \vdash c : \bot\,;\,\Delta}{\Gamma \vdash \#c : B\,;\,\Delta}\ \texttt{W}_i$$

Peirce typed!

Let

$$\pi := \left(\dfrac{\dfrac{\overline{\Gamma, y : A \vdash y : A\,;\ \beta : B}\ \texttt{ax}}{\Gamma, y : A \vdash [\alpha]\, y : \bot\,;\ \alpha : A, \beta : B}\ \texttt{der}}{\Gamma \vdash \mu(y, \beta).\, [\alpha]\, y : A \to_c B\,;\ \alpha : A}\ \texttt{I-}{\to_c} \right)$$

where $\Gamma = x : (A \to_c B) \to_c A$.

Peirce typed!

Let

$$\pi := \left(\dfrac{\dfrac{\overline{\Gamma, y : A \vdash y : A\,;\, \beta : B}\;\texttt{ax}}{\Gamma, y : A \vdash [\alpha]\, y : \bot\,;\, \alpha : A, \beta : B}\;\texttt{der}}{\Gamma \vdash \mu(y,\beta).\,[\alpha]\, y : A \to_c B\,;\, \alpha : A}\;\texttt{I-}{\to_c} \right)$$

where $\Gamma = x : (A \to_c B) \to_c A$.

Then

$$\dfrac{\dfrac{\overline{\Gamma \vdash x : (A \to_c B) \to_c A\,;\, \alpha : A}\;\texttt{ax} \qquad \begin{array}{c}\vdots\\ \pi\end{array} \qquad \dfrac{\overline{\Gamma, y : A \vdash y : A\,;\, \cdot}\;\texttt{ax}}{\Gamma, y : A \vdash [\alpha]\, y : \bot\,;\, \alpha : A}\;\texttt{der}}{\Gamma \vdash x\,[\mu(y,\beta).\,[\alpha]\, y, y.[\alpha]\, y] : \bot\,;\, \alpha : A}\;\texttt{E-}{\to_c}}{\emptyset \vdash \mu(x,\alpha).\, x\,[\mu(y,\beta).\,[\alpha]\, y, y.[\alpha]\, y] : ((A \to_c B) \to_c A) \to_c A\,;\, \cdot}\;\texttt{I-}{\to_c}$$

Peirce typed!

Let

$$\pi := \left(\dfrac{\dfrac{\dfrac{}{\Gamma, y : A \vdash y : A \,;\, \beta : B}\,\texttt{ax}}{\Gamma, y : A \vdash [\alpha]\, y : \bot \,;\, \alpha : A, \beta : B}\,\texttt{der}}{\dfrac{\Gamma, y : A \vdash \#[\alpha]\, y : B \,;\, \alpha : A, \beta : B}{\Gamma \vdash \lambda y. \#[\alpha]\, y : A \to_i B \,;\, \alpha : A}\,\texttt{I-}{\to_i}}\,\texttt{W}_i \right)$$

where $\Gamma = x : (A \to_i B) \to_i A$.

Peirce typed!

Let

$$\pi := \left(\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, y : A \vdash y : A \,;\, \beta : B}\,\texttt{ax}}{\Gamma, y : A \vdash [\alpha]\, y : \bot \,;\, \alpha : A, \beta : B}\,\texttt{der}}{\Gamma, y : A \vdash \#[\alpha]\, y : B \,;\, \alpha : A, \beta : B}\,\texttt{W}_i}{\Gamma \vdash \lambda y.\, \#[\alpha]\, y : A \to_i B \,;\, \alpha : A}\,\texttt{I-}{\to_i} \right)$$

where $\Gamma = x : (A \to_i B) \to_i A$.

Then

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : (A \to_i B) \to_i A \,;\, \alpha : A}\,\texttt{ax} \qquad \begin{array}{c}\vdots\\ \pi\end{array} \qquad \dfrac{\dfrac{}{\Gamma, y : A \vdash y : A \,;\, \cdot}\,\texttt{ax}}{\Gamma, y : A \vdash [\alpha]\, y : \bot \,;\, \alpha : A}\,\texttt{der}}{\Gamma \vdash x\, [\lambda y.\, \#[\alpha]\, y, y.[\alpha]\, y] : \bot \,;\, \alpha : A}\,\texttt{E-}{\to_c}}{\emptyset \vdash \mu(x, \alpha).\, x\, [\lambda y.\, \#[\alpha]\, y, y.[\alpha]\, y] : ((A \to_i B) \to_i A) \to_c A \,;\, \cdot}\,\texttt{I-}{\to_c}$$

Outline

What is Modal Logic?

Carlos ______________________ handsome.

Classical logic: truth

Carlos ________is________ handsome.

Classical logic: truth

Carlos ______*is not*______ handsome.

Modal logic: qualifies truth

Carlos ___*is necessarily*___ handsome.

Modal logic: qualifies truth

Carlos ~~is necessarily~~ ________ handsome.

possibly

Modal logic: qualifies truth

Carlos ______is necessarily______ handsome.
possibly

alethic interpretation

Modal logic: qualifies truth

Carlos __is known to be__ handsome.

Modal logic: qualifies truth

Carlos __is known to be__ handsome. (by me)

epistemic interpretation

Modal logic: qualifies truth

Carlos ___is believed to be___ handsome. (by me)

doxastic interpretation

Modal logic: qualifies truth

Carlos ___is obliged to be___ handsome.

Modal logic: qualifies truth

Carlos is obliged to be handsome.

permission

prohibition

deontic interpretation

What is Modal Logic?

Modal logic: qualifies truth

Carlos ____is now____ handsome.

Modal logic: qualifies truth

Carlos __________ is now __________ handsome.

will be

temporal interpretation

Alethic interpretation

Carlos ___is necessarily___ handsome.

Alethic interpretation

necessarily Carlos is ________________ handsome.

Alethic interpretation

p = Carlos is handsome

necessarily p

Alethic interpretation

p = Carlos is handsome

$$\Box p$$

Alethic interpretation

Carlos ~~___~~ is possibly ~~___~~ handsome.

Alethic interpretation

possibly Carlos is ________________ handsome.

Alethic interpretation

p = Carlos is handsome

possibly p

Alethic interpretation

p = Carlos *is* handsome

$$\Diamond p$$

Truth table

A	B	$A \to B$
1	1	1
1	0	0
0	1	1
0	0	1

Truth table

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Truth tables

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Generalizing

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

v

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Adding relations

v

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Relational models

Adding relations

w

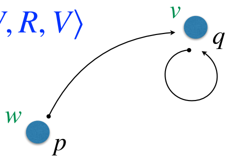
p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

v

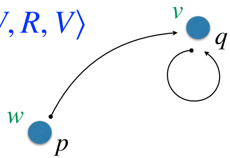
p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Relational models

$\mathscr{M} = \langle W, R, V \rangle$

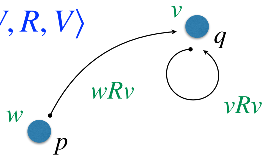


$\mathscr{M} = \langle W, R, V \rangle$



W is a non-empty set of possible worlds.

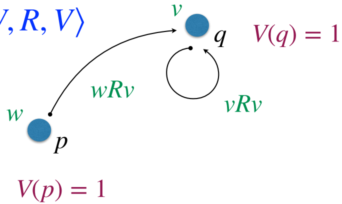
$\mathscr{M} = \langle W, R, V \rangle$



R is the relative accessibility relation:
from the point of view of w, v is possible.

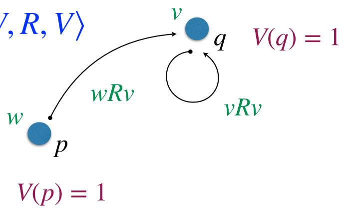
Relational models

$\mathscr{M} = \langle W, R, V \rangle$



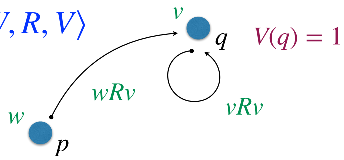
V assigns a truth value to a propositional variable at a world.

$\mathcal{M} = \langle W, R, V \rangle$



For non-atomic propositional formulas:
Just check the truth table
in each world!

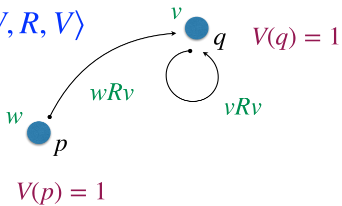
$\mathscr{M} = \langle W, R, V \rangle$



$V(p) = 1$

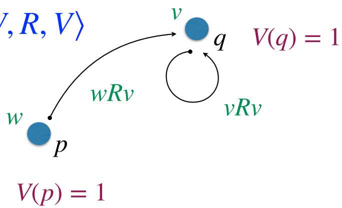
$\mathscr{M}, w \nvDash p \rightarrow q$ $\qquad$ $\mathscr{M}, v \vDash p \rightarrow q$

$$\mathscr{M} = \langle W, R, V \rangle$$



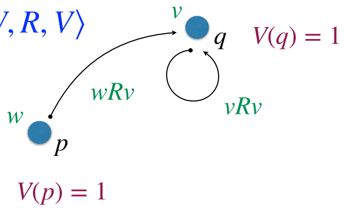
How about modal formulas?

$\mathscr{M} = \langle W, R, V \rangle$



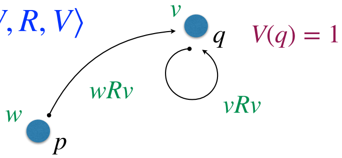
A is *necessary at a world* u provided A is true at every possible world from u.

$\mathscr{M} = \langle W, R, V \rangle$



A is *possible at a world* u provided A is true at some possible world from u.

$\mathscr{M} = \langle W, R, V \rangle$



$V(p) = 1$

$\mathscr{M}, w \nvDash \Box p$	$\mathscr{M}, v \nvDash \Box p$
$\mathscr{M}, w \vDash \Box q$	$\mathscr{M}, v \vDash \Box q$
$\mathscr{M}, w \vDash \Box (p \to q)$	$\mathscr{M}, v \vDash \Box (p \to q)$

$\mathcal{M}, w \Vdash p$	iff	$p \in V(w)$;
$\mathcal{M}, w \Vdash \bot$		never holds;
$\mathcal{M}, w \Vdash \neg A$	iff	$\mathcal{M}, w \nVdash A$;
$\mathcal{M}, w \Vdash A \wedge B$	iff	$\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \vee B$	iff	$\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \rightarrow B$	iff	$\mathcal{M}, w \nVdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash \Box A$	iff	for all v. wRv implies $\mathcal{M}, v \Vdash A$;
$\mathcal{M}, w \Vdash \Diamond A$	iff	there exists v. wRv and $\mathcal{M}, v \Vdash A$.

$\mathcal{M}, w \Vdash p$	iff	$p \in V(w)$;
$\mathcal{M}, w \Vdash \bot$		never holds;
$\mathcal{M}, w \Vdash \neg A$	iff	for all $v.\, w \leq v.\, \mathcal{M}, v \nVdash A$;
$\mathcal{M}, w \Vdash A \wedge B$	iff	$\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \vee B$	iff	$\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \to B$	iff	for all $v.\, w \leq v.\, \mathcal{M}, v \Vdash A$ implies $\mathcal{M}, v \Vdash B$.

Outline

Classical Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$
- **Duality** by De Morgan laws and $\neg\Box A = \Diamond\neg A$
- Axioms: **classical** propositional logic and

$$\mathsf{k}\colon \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \rightarrow B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Relational structures (W, R)

  a non-empty set W of worlds;

  a binary relation $R \subseteq W \times W$;

Intuitionistic Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$
- Independence of the modalities
- Axioms: intuitionistic propositional logic and

$$\mathsf{k}\colon \Box(A \to B) \to (\Box A \to \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Birelational structures $(W, R, \leq)$

  a non-empty set W of worlds;

  a binary relation $R \subseteq W \times W$;

  a preorder $\leq$ on W.

$$(F_1)\quad \begin{tikzcd} u' \arrow[r, dotted, dash, "R"] \arrow[d, dotted, dash, "\leq"'] & v' \arrow[d, dash, "\leq"] \\ u \arrow[r, dash, "R"] & v \end{tikzcd}$$

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$
- Independence of the modalities
- Axioms: intuitionistic propositional logic and

$$\begin{aligned} \mathsf{k}_1 &: \Box(A \to B) \to (\Box A \to \Box B) \\ \mathsf{k}_2 &: \Box(A \to B) \to (\Diamond A \to \Diamond B) \end{aligned}$$

CK (Fitch 1948)

- Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Birelational structures $(W, R, \leq)$

  a non-empty set W of worlds;

  a binary relation $R \subseteq W \times W$;

  a preorder $\leq$ on W.

(F_1)
$$\begin{tikzcd} u' \arrow[r, dotted, dash, "R"] \arrow[d, dotted, dash, "\leq"'] & v' \arrow[d, dash, "\leq"] \\ u \arrow[r, dash, "R"] & v \end{tikzcd}$$

Intuitionistic Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$
- Independence of the modalities
- Axioms: intuitionistic propositional logic and

$$
\begin{aligned}
\mathsf{k}_1 &: \Box(A \to B) \to (\Box A \to \Box B)\\
\mathsf{k}_2 &: \Box(A \to B) \to (\Diamond A \to \Diamond B)\\
\mathsf{k}_3 &: \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)\\
\\
\mathsf{k}_5 &: \neg\Diamond\bot
\end{aligned}
$$

- Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Birelational structures $(W, R, \leq)$

  a non-empty set W of worlds;

  a binary relation $R \subseteq W \times W$;

  a preorder $\leq$ on W.

$$
(F_1)\quad
\begin{tikzcd}
u' \arrow[r, dotted, dash, "R"] & v' \\
u \arrow[u, dotted, dash, "\leq"] \arrow[r, dash, "R"] & v \arrow[u, dash, "\leq"']
\end{tikzcd}
\qquad
(F_2)\quad
\begin{tikzcd}
u' \arrow[r, dotted, dash, "R"] & v' \\
u \arrow[u, dash, "\leq"] \arrow[r, dash, "R"] & v \arrow[u, dotted, dash, "\leq"']
\end{tikzcd}
$$

Intuitionistic Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$
- Independence of the modalities
- Axioms: intuitionistic propositional logic and

$$\begin{array}{l}
\mathrm{k}_1 : \Box(A \to B) \to (\Box A \to \Box B) \\
\mathrm{k}_2 : \Box(A \to B) \to (\Diamond A \to \Diamond B) \\
\mathrm{k}_3 : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B) \\
\mathrm{k}_4 : (\Diamond A \to \Box B) \to \Box(A \to B) \\
\mathrm{k}_5 : \neg \Diamond \bot
\end{array}$$

IK (Plotkin and Stirling 1986)

- Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Birelational structures $(W, R, \leq)$

  a non-empty set W of worlds;

  a binary relation $R \subseteq W \times W$;

  a preorder $\leq$ on W.

$$(F_1)\quad \begin{tikzcd} u' \arrow[r, dash, dotted, "R"] \arrow[d, dash, dotted, "\leq"'] & v' \arrow[d, dash, "\leq"] \\ u \arrow[r, dash, "R"] & v \end{tikzcd} \qquad (F_2)\quad \begin{tikzcd} u' \arrow[r, dash, dotted, "R"] \arrow[d, dash, "\leq"'] & v' \arrow[d, dash, dotted, "\leq"] \\ u \arrow[r, dash, "R"] & v \end{tikzcd}$$

$$x \vDash \Box A \Leftrightarrow \forall y, z.\ \text{if } x \leq y \ \&\ yRz \ \text{ then } \ z \vDash A$$

Intuitionistic Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$
- Independence of the modalities
- Axioms: intuitionistic propositional logic and

$$
\begin{aligned}
\mathsf{k}_1 &: \Box(A \to B) \to (\Box A \to \Box B)\\
\mathsf{k}_2 &: \Box(A \to B) \to (\Diamond A \to \Diamond B)\\
\mathsf{k}_3 &: \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)\\
\mathsf{k}_4 &: (\Diamond A \to \Box B) \to \Box(A \to B)\\
\mathsf{k}_5 &: \neg\Diamond\bot
\end{aligned}
$$

- Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Birelational structures $(W, R, \leq)$

  a non-empty set W of worlds;

  a binary relation $R \subseteq W \times W$;

  a preorder $\leq$ on W.

(F_1)
$$
\begin{tikzcd}
u' \arrow[r, dotted, no head, "R"] \arrow[d, dotted, no head, "\leq"'] & v' \arrow[d, no head, "\leq"] \\
u \arrow[r, no head, "R"] & v
\end{tikzcd}
$$

(F_2)
$$
\begin{tikzcd}
u' \arrow[r, dotted, no head, "R"] \arrow[d, no head, "\leq"'] & v' \arrow[d, dotted, no head, "\leq"] \\
u \arrow[r, no head, "R"] & v
\end{tikzcd}
$$

$$x \vDash \Box A \Leftrightarrow \forall y, z.\ \text{if } x \leq y \ \&\ yRz \ \text{ then } \ z \vDash A$$

$$x \vDash \Diamond A \Leftrightarrow \exists y. xRy \ \text{ and } \ y \vDash A$$

Axioms: classical propositional logic and

$$\mathsf{k}\colon \Box(A \to B) \to (\Box A \to \Box B)$$

Sequent system: classical sequent calculus and

$$\mathsf{k}_\Box \, \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

Intuitionistic modal proof theory

Axioms: intuitionistic propositional logic and

$$\mathsf{k}\colon\ \Box(A \to B) \to (\Box A \to \Box B)$$

Sequent system: intuitionistic sequent calculus and

$$\mathsf{k}_{\Box}\ \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

Axioms: intuitionistic propositional logic and

$$\begin{aligned} \mathsf{k}_1 &: \Box(A \to B) \to (\Box A \to \Box B) \\ \mathsf{k}_2 &: \Box(A \to B) \to (\Diamond A \to \Diamond B) \end{aligned}$$

Sequent system: intuitionistic sequent calculus and

$$\mathsf{k}_\Box \, \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \mathsf{k}_\Diamond \, \frac{\Gamma, A \vdash B}{\Box\Gamma, \Diamond A \vdash \Diamond B}$$

Intuitionistic modal proof theory

Axioms: intuitionistic propositional logic and

$$\begin{aligned}
\mathsf{k}_1 &: \Box(A \to B) \to (\Box A \to \Box B)\\
\mathsf{k}_2 &: \Box(A \to B) \to (\Diamond A \to \Diamond B)\\
\mathsf{k}_3 &: \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)
\end{aligned}$$

$$\mathsf{k}_5 : \neg\Diamond\bot$$

Sequent system: intuitionistic sequent calculus and

$$\mathsf{k}_\Box \, \frac{\Gamma \vdash \Delta}{\Box\Gamma \vdash \Box\Delta} \qquad \mathsf{k}_\Diamond \, \frac{\Gamma, A \vdash \Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}$$

Intuitionistic modal proof theory

Axioms: intuitionistic propositional logic and

$$
\begin{aligned}
&\mathsf{k}_1 : \Box(A \to B) \to (\Box A \to \Box B) \\
&\mathsf{k}_2 : \Box(A \to B) \to (\Diamond A \to \Diamond B) \\
&\mathsf{k}_3 : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B) \\
&\mathsf{k}_4 : (\Diamond A \to \Box B) \to \Box(A \to B) \\
&\mathsf{k}_5 : \neg\Diamond\bot
\end{aligned}
$$

Sequent system: intuitionistic sequent calculus and

$$
\mathsf{k}_\Box \, \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \mathsf{k}_\Diamond \, \frac{\Gamma, A \vdash \Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}
$$

Problem? k_4 is not derivable.

- not a problem for modal type theory...

Intuitionistic modal proof theory

Axioms: intuitionistic propositional logic and

$$\begin{aligned}
&\mathsf{k}_1 : \Box(A \to B) \to (\Box A \to \Box B)\\
&\mathsf{k}_2 : \Box(A \to B) \to (\Diamond A \to \Diamond B)\\
&\mathsf{k}_3 : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)\\
&\mathsf{k}_4 : (\Diamond A \to \Box B) \to \Box(A \to B)\\
&\mathsf{k}_5 : \neg\Diamond\bot
\end{aligned}$$

Sequent system: intuitionistic sequent calculus and

$$\mathsf{k}_\Box\ \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \mathsf{k}_\Diamond\ \frac{\Gamma, A \vdash \Delta}{\Box\Gamma, \Diamond A \vdash \Diamond\Delta}$$

Problem? k_4 is not derivable.

- not a problem for modal type theory...

labeled sequent system: (Simpson 1994)

$$\Box_{\mathsf{L}}\ \frac{xRy, \Gamma, x : \Box A, y : A \Rightarrow z : B}{xRy, \Gamma, x : \Box A \Rightarrow z : B} \qquad \Box_{\mathsf{R}}\ \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A}\ y \text{ is fresh}$$

$$\Diamond_{\mathsf{L}}\ \frac{xRy, \Gamma, y : A \Rightarrow z : B}{\Gamma, x : \Diamond A \Rightarrow z : B}\ y \text{ is fresh} \qquad \Diamond_{\mathsf{R}}\ \frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond A}$$

Outline

Ecumenical modalities

$$[\Box A]_x = \forall y(R(x,y) \to [A]_y) \qquad [\Diamond A]_x = \exists y(R(x,y) \wedge [A]_y)$$

Ecumenical modalities

$$[\Box A]_x = \forall y(R(x,y) \to [A]_y) \qquad [\Diamond A]_x = \exists y(R(x,y) \wedge [A]_y)$$

$$\begin{array}{lll} \mathcal{M}, w \models \Box A & \text{iff} & \text{for all } v \text{ such that } wRv, \mathcal{M}, v \models A. \\ \mathcal{M}, w \models \Diamond A & \text{iff} & \text{there exists } v \text{ such that } wRv \text{ and } \mathcal{M}, v \models A. \end{array}$$

$R(x,y)$ represents the accessibility relation R in a Kripke frame.

$$[\Box A]_x = \forall y(R(x,y) \to [A]_y) \qquad [\Diamond A]_x = \exists y(R(x,y) \wedge [A]_y)$$

$$\vdash_{OL} A \quad \text{iff} \quad \vdash_{ML} \forall x.[A]_x$$

- ML = classical logic $\rightsquigarrow$ OL = classical modal logic K.
- ML = intuitionistic logic $\rightsquigarrow$ OL = intuitionistic modal logic IK.
- ML = Ecumenical logic $\rightsquigarrow$ OL = Ecumenical modal logic EK.

$$[\Box A]_x^e = \forall y(R(x,y) \rightarrow_i [A]_y^e)$$

$$[\Diamond_i A]_x^e = \exists_i y(R(x,y) \wedge [A]_y^e) \qquad [\Diamond_c A]_x^e = \exists_c y(R(x,y) \wedge [A]_y^e)$$

$$[\Box A]^e_x = \forall y(R(x,y) \to_i [A]^e_y)$$

$$[\Diamond_i A]^e_x = \exists_i y(R(x,y) \wedge [A]^e_y) \qquad [\Diamond_c A]^e_x = \exists_c y(R(x,y) \wedge [A]^e_y)$$

- $\Diamond_c A \Leftrightarrow_i \neg\Box\neg A$ but $\Diamond_i A \not\Leftrightarrow_i \neg\Box\neg A$.
- Restricted to the classical fragment: $\Box$ and $\Diamond_c$ are duals.

Ecumenical Modal Logic

- Formulas: $A ::= p_i \mid p_c \mid \bot \mid A \wedge A \mid A \vee_i A \mid A \vee_c A \mid A \rightarrow_i A \mid A \rightarrow_c A \mid \Box A \mid \Diamond_i A \mid \Diamond_c A$
- Independence of the modalities
- Axioms: ecumenical propositional logic and

$$
\begin{array}{l}
\mathrm{k}_1 : \Box(A \rightarrow_i B) \rightarrow_i (\Box A \rightarrow_i \Box B) \\
\mathrm{k}_2 : \Box(A \rightarrow_i B) \rightarrow_i (\Diamond_i A \rightarrow_i \Diamond_i B) \\
\mathrm{k}_3 : \Diamond_i(A \vee_i B) \rightarrow_i (\Diamond A \vee_i \Diamond B) \\
\mathrm{k}_4 : (\Diamond_i A \rightarrow_i \Box B) \rightarrow_i \Box(A \rightarrow_i B) \\
\mathrm{k}_5 : \neg \Diamond_i \bot
\end{array}
$$

EK (Marin et al. 2020)

- Rules: modus ponens: $\dfrac{A \quad A \rightarrow B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Ecumenical Birelational structures $(W, R, \leq)$

  a non-empty set W of worlds;

  a binary relation $R \subseteq W \times W$;

  a preorder $\leq$ on W.

$$
(F_1)\quad
\begin{tikzcd}
u' \arrow[r, dash, dotted, "R"] \arrow[d, dash, dotted, "\leq"'] & v' \arrow[d, dash, "\leq"] \\
u \arrow[r, dash, "R"] & v
\end{tikzcd}
\qquad
(F_2)\quad
\begin{tikzcd}
u' \arrow[r, dash, dotted, "R"] \arrow[d, dash, "\leq"'] & v' \arrow[d, dash, dotted, "\leq"] \\
u \arrow[r, dash, "R"] & v
\end{tikzcd}
$$

Ecumenical Modal Logic

- Formulas: $A ::= p_i \mid p_c \mid \bot \mid A \wedge A \mid A \vee_i A \mid A \vee_c A \mid A \rightarrow_i A \mid A \rightarrow_c A \mid \Box A \mid \Diamond_i A \mid \Diamond_c A$
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\begin{array}{l}
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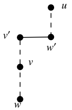

$$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A \text{ iff } \forall v \geq w. \exists u. v\, (\leq \circ R \circ \leq)\, u,\ \mathcal{M}, u \models_{\mathsf{E}} A$$

Labeled modal rules:

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A}\ \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A}\ \Diamond_i R$$

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$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A}\ \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A}\ \Diamond_i R$$

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Ecumenical modal proof theory

Labeled modal rules:

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Extensions:

Axiom	Condition	First-Order Formula
T : $\Box A \rightarrow_i A \wedge A \rightarrow_i \Diamond_i A$	Reflexivity	$\forall x.R(x,x)$
4 : $\Box A \rightarrow_i \Box\Box A \wedge \Diamond_i \Diamond_i A \rightarrow_i \Diamond_i A$	Transitivity	$\forall x,y,z.(R(x,y) \wedge R(y,z)) \rightarrow_i R(x,z)$
5 : $\Box A \rightarrow_i \Box\Diamond_i A \wedge \Diamond_i \Box A \rightarrow_i \Diamond_i A$	Euclideaness	$\forall x,y,z.(R(x,y) \wedge R(x,z)) \rightarrow_i R(y,z)$
B : $A \rightarrow_i \Box\Diamond_i A \wedge \Diamond_i \Box A \rightarrow_i A$	Symmetry	$\forall x,y.R(x,y) \rightarrow_i R(y,x)$

Ecumenical modal proof theory

Labeled modal rules:

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A}\ \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A}\ \Diamond_i R$$

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Axiom	Condition	First-Order Formula
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$5 : \Box A \rightarrow_i \Box\Diamond_i A \wedge \Diamond_i\Box A \rightarrow_i \Diamond_i A$	Euclideaness	$\forall x,y,z.(R(x,y) \wedge R(x,z)) \rightarrow_i R(y,z)$
$\mathrm{B} : A \rightarrow_i \Box\Diamond_i A \wedge \Diamond_i\Box A \rightarrow_i A$	Symmetry	$\forall x,y.R(x,y) \rightarrow_i R(y,x)$

Rules:

$$\frac{xRx, \Gamma \vdash w : C}{\Gamma \vdash w : C}\ \mathrm{T} \qquad \frac{xRz, \Gamma \vdash w : C}{xRy, yRz, \Gamma \vdash w : C}\ 4$$

$$\frac{yRz, \Gamma \vdash w : C}{xRy, xRz, \Gamma \vdash w : C}\ 5 \qquad \frac{yRx, \Gamma \vdash w : C}{xRy, \Gamma \vdash w : C}\ \mathrm{B}$$

Crossing the fine line!!

Easy to prove: $\vdash_{\mathsf{labEK}} x : \Box A \rightarrow_i \neg \Diamond_i \neg A$.

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Easy to prove: $\vdash_{\mathsf{labEK}} x : \Box A \to_i \neg\Diamond_i\neg A$.

Assume T $+\ \neg\Diamond_i\neg A \to_i \Box A$. Then

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$xRy, y : A, y : \neg(A \vee_i \neg A) \vdash y : A$}
\doubleLine
\UnaryInfC{$xRy, y : A, y : \neg(A \vee_i \neg A) \vdash y : \bot$}
\doubleLine
\UnaryInfC{$xRy, y : \neg(A \vee_i \neg A) \vdash x : \bot$}
\RightLabel{$\Diamond_i L$}
\UnaryInfC{$x : \Diamond_i\neg(A \vee_i \neg A) \vdash x : \bot$}
\RightLabel{$\neg R$}
\UnaryInfC{$\vdash x : \neg\Diamond_i\neg(A \vee_i \neg A)$}
\RightLabel{eq}
\UnaryInfC{$\vdash x : \Box(A \vee_i \neg A)$}
\AxiomC{}
\RightLabel{init}
\UnaryInfC{$xRx, x : (A \vee_i \neg A) \vdash x : A \vee_i \neg A$}
\RightLabel{$\Box L$}
\UnaryInfC{$xRx, x : \Box(A \vee_i \neg A) \vdash x : A \vee_i \neg A$}
\RightLabel{T}
\UnaryInfC{$x : \Box(A \vee_i \neg A) \vdash x : A \vee_i \neg A$}
\RightLabel{cut}
\BinaryInfC{$\vdash x : A \vee_i \neg A$}
\end{prooftree}
$$

Crossing the fine line!!

Easy to prove: $\vdash_{\mathsf{labEK}} x : \Box A \to_i \neg\Diamond_i\neg A$.

Assume $\mathsf{T} + \neg\Diamond_i\neg A \to_i \Box A$. Then

$$
\begin{prooftree}
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\UnaryInfC{$xRy, y : A, y : \neg(A \vee_i \neg A) \vdash y : A$}
\doubleLine
\UnaryInfC{$xRy, y : A, y : \neg(A \vee_i \neg A) \vdash y : \bot$}
\doubleLine
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\RightLabel{cut}
\BinaryInfC{$\vdash x : A \vee_i \neg A$}
\end{prooftree}
$$

Outline

Getting rid of negation

$\boxed{\text{LE}}$

$\Gamma, \neg\Delta \vdash C$

Getting rid of negation

$\boxed{\text{LE}} \longrightarrow \boxed{\text{LCE}}$

$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$

Getting rid of negation

$\boxed{\text{LE}}$ $\longrightarrow$ $\boxed{\text{LCE}}$

$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta ; C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to_i B} \to_i R$$

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$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \rightarrow_c B} \rightarrow_c R$$

$\boxed{\text{LE}} \longrightarrow \boxed{\text{LCE}}$

$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$$

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$\boxed{\text{labEK}}$

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A} \Diamond_c R$$

$\boxed{\text{LE}} \longrightarrow \boxed{\text{LCE}}$

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$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R \qquad\qquad \frac{\Gamma, A \vdash B, \Delta; \cdot}{\Gamma \vdash A \to_c B, \Delta; \cdot} \to_c R$$

$\boxed{\text{labEK}} \longrightarrow \boxed{\text{Pure labEK}}$

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A} \Diamond_c R \qquad\qquad \frac{xRy, \Gamma \vdash y : A, x : \Diamond_c A, \Delta; \cdot}{xRy, \Gamma \vdash x : \Diamond_c A, \Delta; \cdot} \Diamond_c R$$

Outline

- Real Ecumenical Mathematics!!! We know that if all operators have a constructive "reading", the axiom of choice is a theorem in Martin-Löf Type Theory. But what would happen if we have hybrid readings of these same operators?

The ecumenical future

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  - Barroso-Nascimento has an ecumenical system for intuitionistic and minimal logic;
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  - Rasga and Sernadas: how to systematically connect translations to ecumenical systems and propose an ecumenical system for classical logic and Jaskowski's paraconsistent logic.

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- etc!!!

End of the talk

Obrigada!!!

Gracias!!!

Taing mhòr!!!

☺