Intuitionistic modalities through proof theory

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joint work with A. Das, I. van der Giessen, M. Girlando, R. Kuznets, M. Morales, and L. Straßburger

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Intuitionistic modal proof theory

Modal type theory is a flavor of type theory with rules for modalities, hence type theory which on propositions reduces to modal logic [nLab]

constructive (somehow)

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(^ constructive (somehow))

Work on constructive modal logic (starts with) Curry-Howard: λ-calculi that arose as byproducts of the proof theory of modal logic and their associated computational & categorical interpretations [Kavvos 2016]

Intuitionistic modal proof theory

- sequent calculus & axiomatic presentations relatively well-known
- natural deduction more controversial

→ "extended Curry-Howard isomorphism"

[Bellin, de Paiva, Ritter 2001]

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$$\frac{}{\Gamma, A \Rightarrow A}\,ax \qquad \frac{}{\Gamma, \bot \Rightarrow C}\,\bot\text{-}l$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\to\text{-}r \qquad \frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\,\to\text{-}l$$

$$\frac{\Gamma \Rightarrow B}{\Box\Gamma \Rightarrow \Box B}\,\Box_k \qquad \frac{\Gamma, A \Rightarrow B}{\Box\Gamma, \Diamond A \Rightarrow \Diamond B}\,\Diamond_k$$

[Wijesekera 1990]

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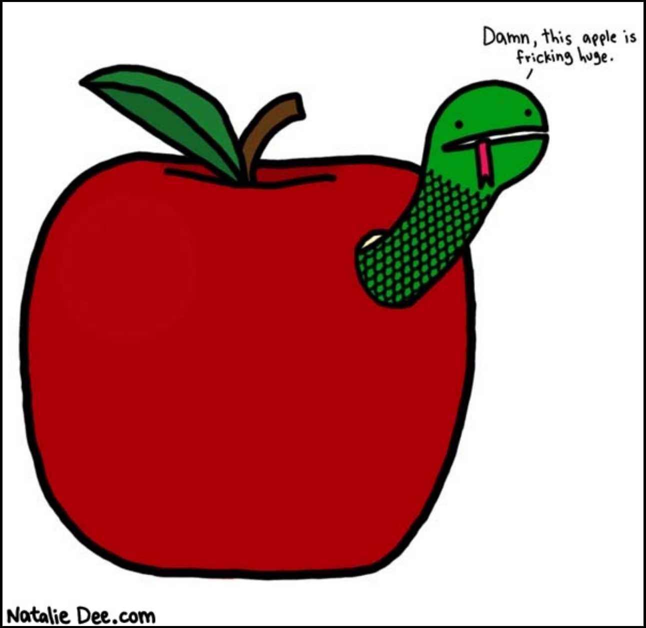
k_1: $\Box(A \to B) \to (\Box A \to \Box B)$

k_2: $\Box(A \to B) \to (\Diamond A \to \Diamond B)$

$$\frac{A \to B \qquad A}{B}\,mp \qquad\qquad \frac{A}{\Box A}\,nec$$


Damn, this apple is fricking huge.
Natalie Dee.com

Intuitionistic modal proof theory

- sequent calculus & axiomatic presentations relatively well-known
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→ "extended Curry-Howard isomorphism"

[Bellin, de Paiva, Ritter 2001]

our logic comes from categorical logic

In fact, it is the logical isolate of a multimodal MLTT

[Kavvos & Gratzer 2023]

Relatively well-known axiomatic systems & sequent calculi

Intuitionistic modal logic:

Intuitionistic propositional logic

$$nec \ \frac{A}{\Box A}$$

$$k_1 : \Box(A \to B) \to (\Box A \to \Box B)$$

$$mp \ \frac{A \quad A \to B}{B}$$

iK

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$CK = k_1 + k_2$

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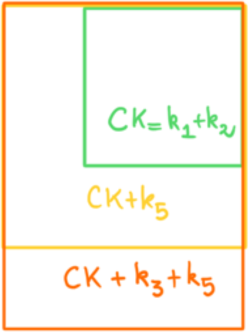

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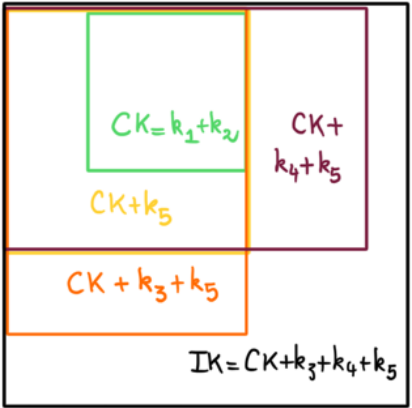


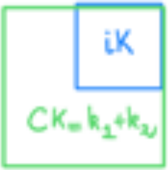
Sequent calculi for intuitionistic modal logic:

iK

$\mathsf{k}_1 : \Box(A \to B) \to (\Box A \to \Box B)$

$$\mathsf{k}_\Box \; \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

Sequent calculi for intuitionistic modal logic:



$$\mathsf{k}_1 : \Box(A \to B) \to (\Box A \to \Box B) \qquad \mathsf{k}_\Box \; \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

$$\mathsf{k}_2 : \Box(A \to B) \to (\Diamond A \to \Diamond B) \qquad \mathsf{k}_\Diamond \; \frac{\Gamma, B \Rightarrow A}{\Box\Gamma, \Diamond B \Rightarrow \Diamond A}$$

Sequent calculi for intuitionistic modal logic:

$\mathsf{k}_1 : \Box(A \to B) \to (\Box A \to \Box B)$ $\qquad \mathsf{k}_\Box \; \dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$

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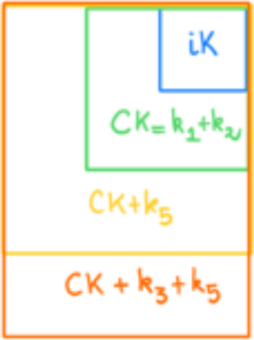
$\mathsf{k}_5 : \Diamond\bot \to \bot$ $\qquad \mathsf{k}_\bot \; \dfrac{\Gamma, B \Rightarrow}{\Box\Gamma, \Diamond B \Rightarrow}$

Sequent calculi for intuitionistic modal logic:



$k_1 : \Box(A \to B) \to (\Box A \to \Box B)$

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$$k_\Diamond \ \frac{\Gamma, B \Rightarrow A}{\Box\Gamma, \Diamond B \Rightarrow \Diamond A}$$

$k_5 : \Diamond\bot \to \bot$

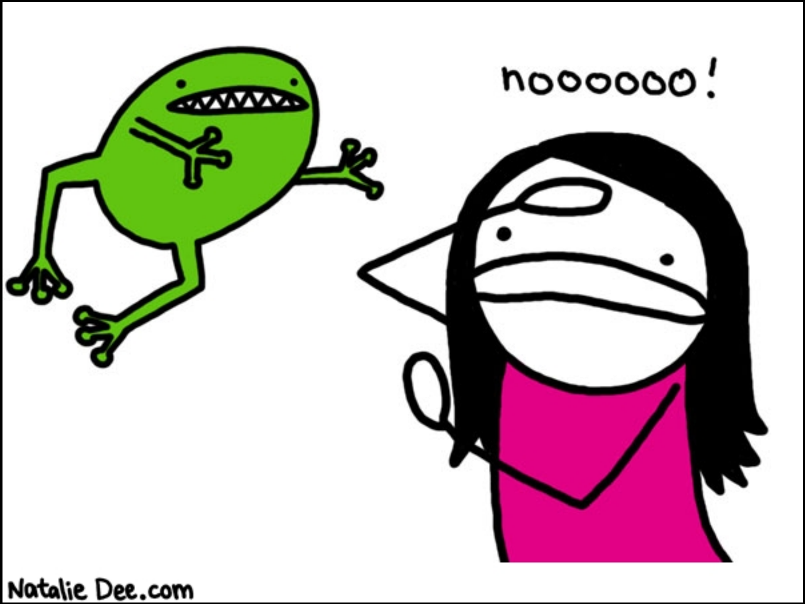
$$k_\bot \ \frac{\Gamma, B \Rightarrow}{\Box\Gamma, \Diamond B \Rightarrow}$$

$k_3 : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

$$\star \ \frac{\Gamma, B \Rightarrow \Delta}{\Box\Gamma, \Diamond B \Rightarrow \Diamond\Delta}$$

Not cut-free complete:

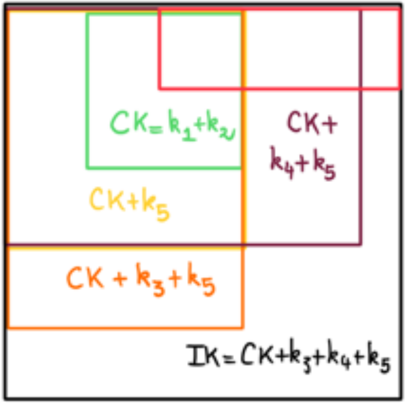
$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\UnaryInfC{$B \to C \Rightarrow B \to C$}
\BinaryInfC{$A \vee (B \to C) \Rightarrow A, B \to C$}
\LeftLabel{$\star$}
\UnaryInfC{$\Diamond(A \vee (B \to C)) \Rightarrow \Diamond A, \Diamond(B \to C)$}
\AxiomC{}
\UnaryInfC{$B \Rightarrow B$}
\AxiomC{}
\UnaryInfC{$C \Rightarrow C$}
\BinaryInfC{$B \to C, B \Rightarrow C$}
\LeftLabel{$\mathsf{k}_\Diamond$}
\UnaryInfC{$\Diamond(B \to C), \Box B \Rightarrow \Diamond C$}
\LeftLabel{cut}
\BinaryInfC{$\Diamond(A \vee (B \to C)), \Box B \Rightarrow \Diamond A, \Diamond C$}
\UnaryInfC{$\Diamond(A \vee (B \to C)) \Rightarrow \Diamond A, \Box B \to \Diamond C$}
\UnaryInfC{$\Diamond(A \vee (B \to C)) \Rightarrow \Diamond A \vee (\Box B \to \Diamond C)$}
\UnaryInfC{$\Rightarrow \Diamond(A \vee (B \to C)) \to (\Diamond A \vee (\Box B \to \Diamond C))$}
\end{prooftree}
$$


nooooooo!
Natalie Dee.com

Why should we even look at diamonds?

Claimed that they all coincide on their ◇-free fragments!

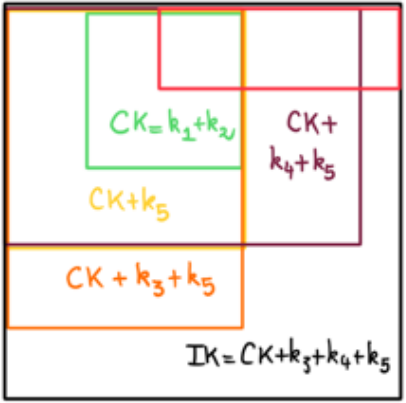
$A ::= p \mid A \wedge A \mid A \vee A \mid \bot \mid A \rightarrow A \mid \Box A$



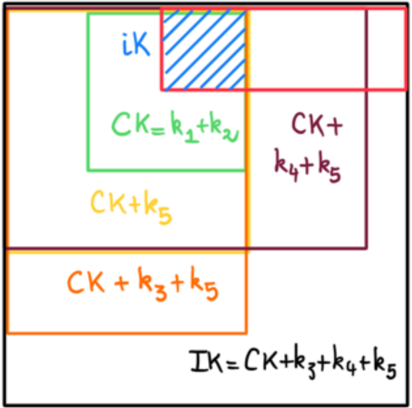
Claimed that they all coincide on their $\Diamond$-free fragments!

$A ::= p \mid A \wedge A \mid A \vee A \mid \bot \mid A \rightarrow A \mid \Box A$

L_1 is $\Box$-conservative over L_2 if they have the same $\Diamond$-free theorems.

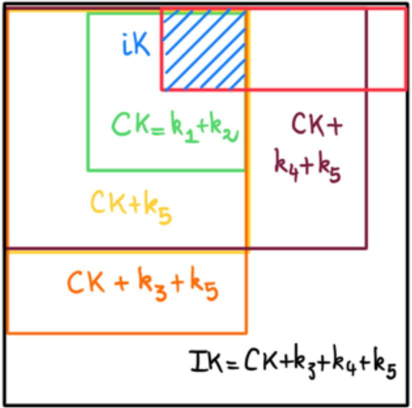


Result 1: $\mathsf{CK} + \mathsf{k}_3 + \mathsf{k}_4$ is $\Box$-conservative over $i\mathsf{K}$.



Result 1: $\mathsf{CK} + \mathsf{k}_3 + \mathsf{k}_4$ is $\Box$-conservative over $i\mathsf{K}$.

Result 2: $\mathsf{CK} + \mathsf{k}_3 + \mathsf{k}_5$ is $\Box$-conservative over $i\mathsf{K}$.



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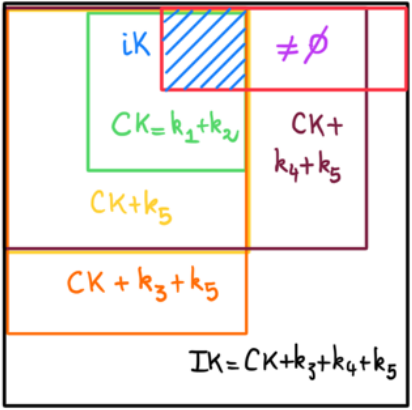
Result 3: $\mathsf{CK} + \mathsf{k}_4 + \mathsf{k}_5$ is not □-conservative over $i\mathsf{K}$.



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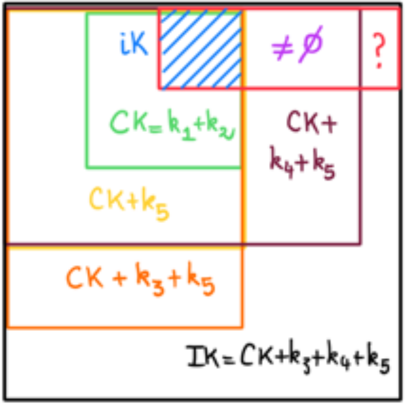
Result 3: $\mathsf{CK} + \mathsf{k}_4 + \mathsf{k}_5$ is not $\Box$-conservative over $i\mathsf{K}$.

Open question: Is IK $\Box$-conservative over $\mathsf{CK} + \mathsf{k}_4 + \mathsf{k}_5$?



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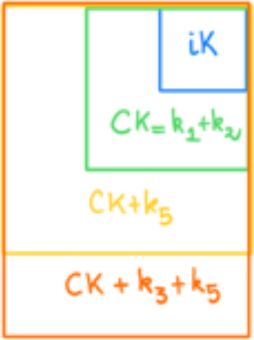

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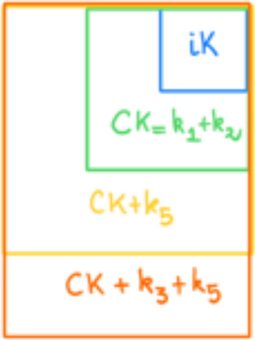
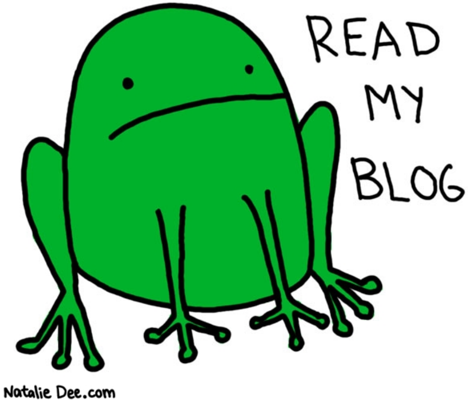
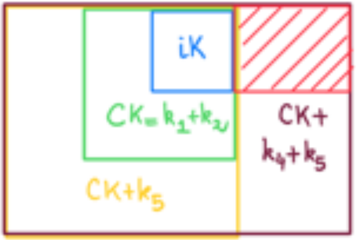
As a simple translation of the axioms, either $\Diamond A \rightsquigarrow \top$ or $\Diamond A \rightsquigarrow \bot$.





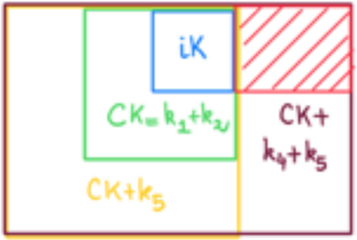
https://prooftheory.blog/2024/03/20/a-note-on-conservativity-in-constructive-modal-logics/

Result 3: $\mathsf{CK} + \mathsf{k}_4 + \mathsf{k}_5$ is not $\Box$-conservative over $i\mathsf{K}$.



Result 3: $\mathsf{CK} + \mathsf{k}_4 + \mathsf{k}_5$ is not $\Box$-conservative over $i\mathsf{K}$.

As a corollary of a Gödel-Gentzen negative translation for $\mathsf{CK} + \mathsf{k}_4 + \mathsf{k}_5$.



Gödel-Gentzen negative translation

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For modal formula A, define another modal formula A^N:

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$$\begin{aligned}
\bot^N &:= \bot \\
p^N &:= \neg\neg p \\
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(A \wedge B)^N &:= A^N \wedge B^N \\
(A \to B)^N &:= A^N \to B^N \\
(\Diamond A)^N &:= \neg\Box\neg A^N \\
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Theorem: If $\mathsf{K} \vdash A$ then $\mathsf{CK} + k_4 + k_5 \vdash A^N$

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Theorem: If $\mathsf{K} \vdash A$ then $\mathsf{CK} + k_4 + k_5 \vdash A^N$

In particular: $\mathsf{CK} + \mathsf{k}_4 + \mathsf{k}_5 \vdash \neg\neg\Box\bot \to \Box\bot$ but $i\mathsf{K} \nvdash \neg\neg\Box\bot \to \Box\bot$

[Das & M. 2023]

Modalities need structure

nested sequents

dual context

Modalities need structure

locks & keys

prefixed tableaux


hop!
hop!
hop!
Natalie Dee.com

Modal logic

$$A ::= p \in At \mid A \wedge A \mid A \vee A \mid \bot \mid A \to A \mid \Box A \mid \Diamond A$$

$$A ::= p \in At \mid A \wedge A \mid A \vee A \mid \bot \mid A \rightarrow A \mid \Box A \mid \Diamond A$$

Relational semantics: (W, R, V)
Set of worlds W with an arbitrary relation R on W and an arbitrary valuation $V : W \rightarrow \wp(At)$

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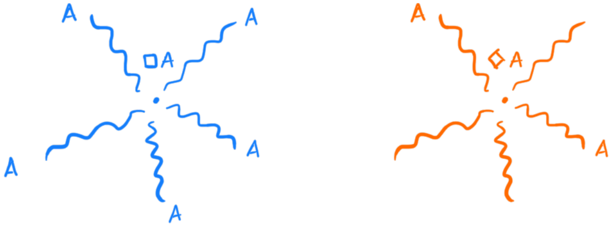
$w \Vdash \Box A \iff$ for all v s.t. wRv : $v \Vdash A$
$w \Vdash \Diamond A \iff$ there exists v s.t. wRv and $v \Vdash A$



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In classical modal logic: $\Diamond A \equiv \neg\Box\neg A$ with $\neg A := A \to \bot$.

Modal logic

Labelled sequents: [Negri, 2005]

A labelled sequent $\mathcal{G}$ is a triple $\mathcal{R}, \Gamma \Longrightarrow \Delta$ with:

- $\mathcal{R}$ set of R-relational atoms xRy
- Γ, Δ multisets of labelled formulas $x{:}A$

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$\Box$ right-rule:

$$\Box\text{-}r \; \frac{\mathcal{R}, xRz, \Gamma \Longrightarrow \Delta, z{:}A}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A} \; z \text{ fresh}$$

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Soundness and completeness:

Modal logic K $\leftrightarrow$ Relational models $\leftrightarrow$ R-labelled sequents

First attempt at constructurizing

Labelled sequents: [Simpson, 1994]

A labelled sequent $\mathcal{G}$ is a triple $\mathcal{R}, \Gamma \Longrightarrow z{:}C$ with:

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$$x{:}\ \Box A := \forall z(xRz \to z{:}\ A)$$

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Intuitionistic modal logic

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Soundness and completeness:

Modal logic IK $\leftrightarrow$ R-labelled sequents

Pro:

- Direct connection with standard translation
- Manageable & easy to extend

IK4

$$\frac{xRz, \mathcal{R}, \Gamma \vdash u : C}{xRy, yRz, \mathcal{R}, \Gamma \vdash u : C}$$

transitive models

$\Box A \to \Box\Box A$

+

IGL

infinite proof trees (underprogress condition)

terminating models

$\Box(\Box A \to A) \to \Box A$

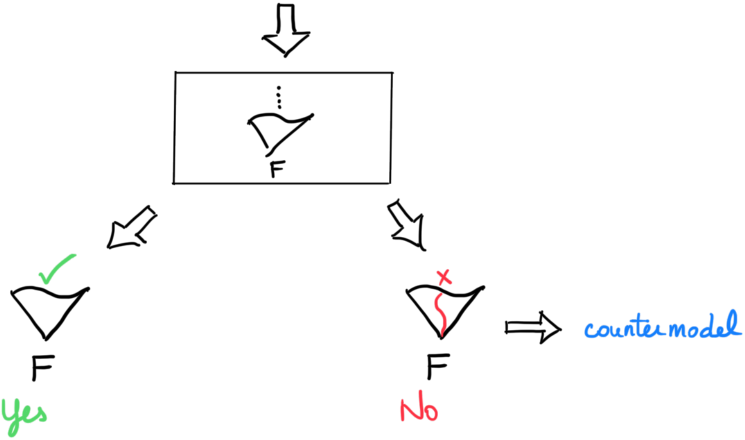
[Das, van der Giessen, M. 2024]

Con:

- No direct connection with semantics

Decision procedure via proof search

is F valid in $\mathcal{L}$?



Intuitionistic logic

$$A ::= p \in At \mid A \wedge A \mid A \vee A \mid \bot \mid A \rightarrow A$$

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Labelled sequents: [Negri, 2012]

A labelled sequent $\mathcal{G}$ is a triple $\mathcal{R}, \Gamma \Longrightarrow \Delta$ with:

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$$\rightarrow\text{-}r \; \frac{\mathcal{R}, x{\leq}z, \Gamma, z{:}A \Longrightarrow \Delta, z{:}B}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \rightarrow B} \; z \text{ fresh}$$

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$$\leq \text{rf} \; \frac{\mathcal{R}, x \leq x, \Gamma \Longrightarrow \Delta}{\mathcal{R}, \Gamma \Longrightarrow \Delta} \qquad \leq \text{tr} \; \frac{\mathcal{R}, x \leq y, y \leq z, x \leq z, \Gamma \Longrightarrow \Delta}{\mathcal{R}, x \leq y, y \leq z, \Gamma \Longrightarrow \Delta}$$

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Soundness and completeness:

Intuitionistic prop. logic $\leftrightarrow$ Preorder semantics $\leftrightarrow$ $\leq$-labelled sequents

Second attempt at constructivizing


Hey, mr. lazy. next time, instead of talking about what other people do, go out and actually do something yourself
Natalie Dee.com

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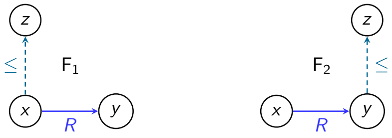

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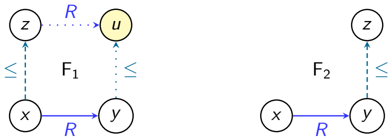

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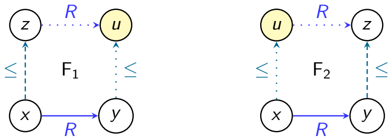

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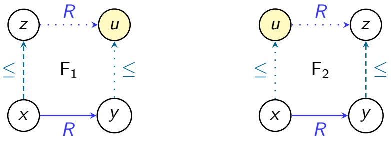

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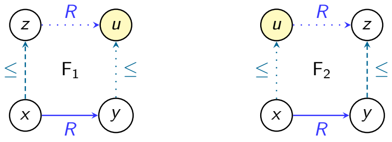

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In intuitionistic modal logic: $\Box A$ and $\Diamond A$ are not dual

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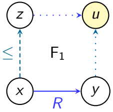

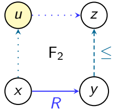

$$\mathsf{F}_1\ \frac{\mathcal{R}, xRy, x \leq z, y \leq u, zRu, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, x \leq z, \Gamma \Longrightarrow \Delta} \qquad \mathsf{F}_2\ \frac{\mathcal{R}, xRy, y \leq z, x \leq u, uRz, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, y \leq z, \Gamma \Longrightarrow \Delta}$$

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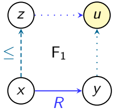

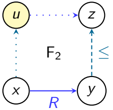

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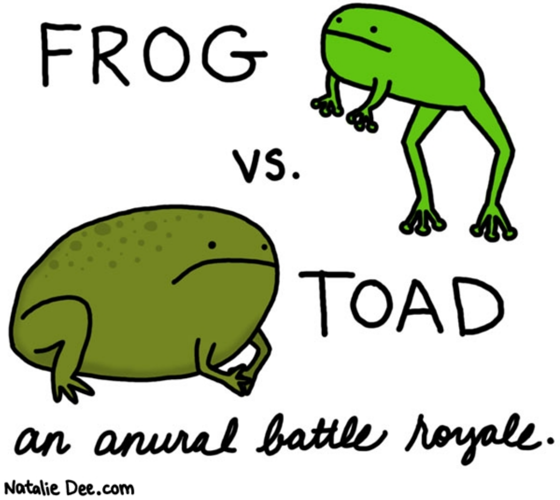
Soundness and completeness:

Logic IK $\leftrightarrow$ Birelational models $\leftrightarrow$ Fully labelled sequents

Pro :

- Finer grained expressivity
- Simple proof search procedure for IK

[Girlando et al 2024]


FROG
VS.
TOAD
an anural battle royale.
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Related Work:

- Gao & Olivetti (& co-authors)
- Straßburger (& co-authors)
- de Groot, Shilito & Clouston / Pacheco


thank you
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