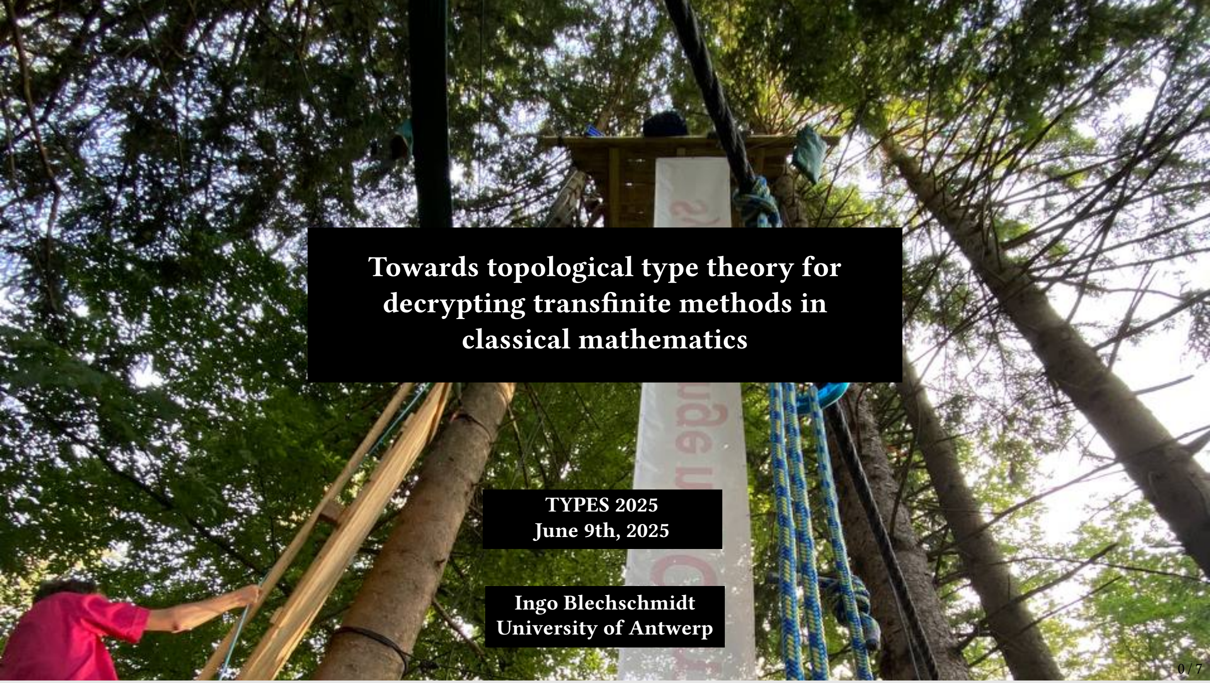


Towards topological type theory for decrypting transfinite methods in classical mathematics

TYPES 2025
June 9th, 2025

Ingo Blechschmidt
University of Antwerp

2025-06-20

Towards topological type theory for decrypting transfinite methods in classical mathematics

A classical logic fairy tale

0 / 7

Narrator. Once upon a time, in a kingdom far, far away, the queen of the country and of all Möbius strips called for her royal philosopher.

Queen. Philosopher! I ask you to carry out the following order. Get me the Philosopher's Stone, or alternatively find out how one could produce arbitrary amounts of gold with it!

Philosopher. But my queen! I haven't studied anything useful! How could I fulfill this order?

Queen. That is not my concern. I'll see you again tomorrow. Should you not accomplish the task, I will take your head off.

Narrator. After a long and wakeful night the philosopher was called to the queen again.

Queen. Tell me! What do you have to report?

Philosopher. It was not easy and I needed to follow lots of obscure references, but finally I actually found out how to use the Philosopher's Stone to produce arbitrary amounts of gold. But only I can conduct this procedure, your royal highness.

Queen. Alright. So be it.

Narrator. And so years passed by, during which the philosopher imagined herself to be safe. The queen searched for the stone on her own, but as long as she hadn't found it, the philosopher didn't need to worry. Yet one day the impossible happened: The queen has found the stone! And promptly called for her philosopher.

Queen. Philosopher, look! I have found the Philosopher's Stone! Now live up to your promise! *[She hands over the stone.]*

Philosopher. Thank you. *[She inspects the stone.]* This is indeed the Philosopher's Stone. Many years ago you asked me to either acquire the Philosopher's Stone or find out how to produce arbitrary amounts of gold using it. Now it's my pleasure to present to you the Philosopher's Stone. *[She returns the stone.]*

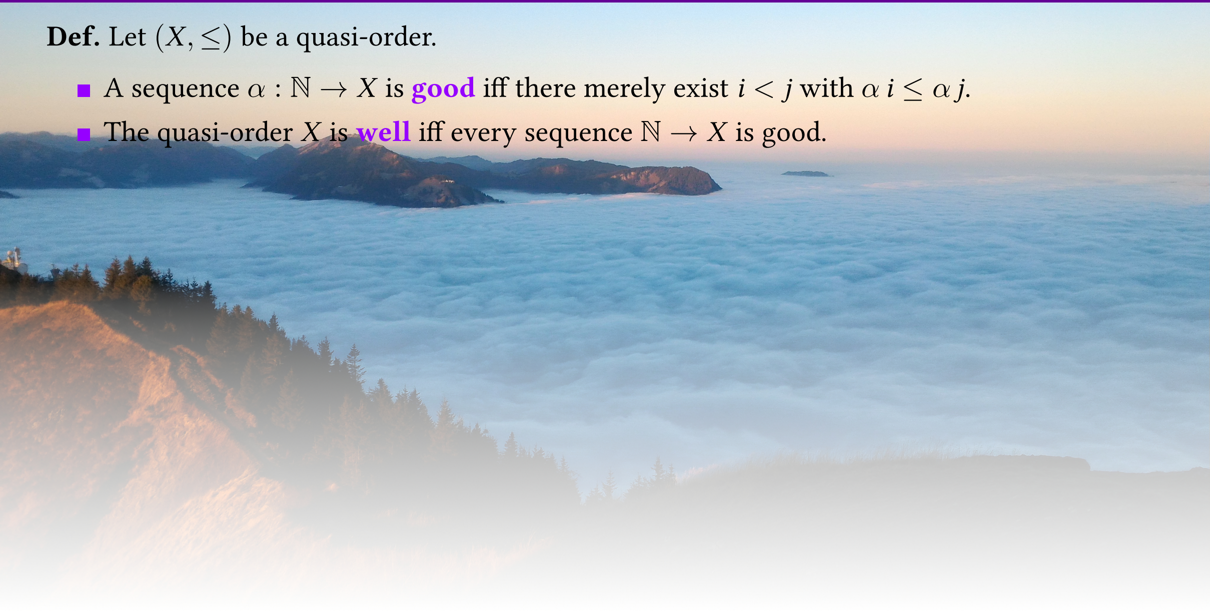
Adapted from Edward Yang's blog.

The idea conveyed in the fairy tale is at the core of the *constructive* proof of $\neg\neg(A \vee \neg A)$. While the law of excluded middle (LEM) is not available in constructive mathematics, up to a double negation every instance of it is.

This observation gives rise to the double negation translation, a transformation of (statements and) proofs in classical logic to proofs in intuitionistic logic, at the only expense of introducing double negations; by following this up with Friedman's trick with the nontrivial exit continuation ("Baby Barr Theorem"), we can in many cases even get rid of these double negations in a second step.

Getting rid of applications of the law of excluded middle is in many contexts (an amazing mind-boggingly wonder at first, but then) a routine matter—see here for more details and pointers to the literature.

A case study in Hilbert’s program



Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well** iff every sequence $\mathbb{N} \to X$ is good.

Well quasi-orders are an important notion in proof theory and termination analysis.

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well** iff every sequence $\mathbb{N} \to X$ is good.

Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. $\square$

offensive?

$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$

$\leq$

The presented proof rests on the law of excluded middle and hence cannot immediately be interpreted as a program for finding suitable indices $i < j$. However, constructive proofs are also possible (for instance by induction on the value of a given term of the sequence, see Constructive combinatorics of Dickson's Lemma by Iosif Petrakis for several fine quantitative results). And even more: There is a procedure for regarding this proof—and many others in the theory of well quasi-orders—as *blueprints* for more informative constructive proofs. This shall be our motto for today:

Do not take classical proofs literally, instead ask which constructive proofs they are blueprints for.

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well** iff every sequence $\mathbb{N} \to X$ is good.

Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. $\square$

offensive?

Key stability results

Assuming LEM and DC, …

Dickson: If X and Y are well, so is $X \times Y$.

Higman: If X is well, so is List X.

Kruskal: If X is well, so is Tree X.

$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$

$\leq$

The presented proof rests on the law of excluded middle and hence cannot immediately be interpreted as a program for finding suitable indices $i < j$. However, constructive proofs are also possible (for instance by induction on the value of a given term of the sequence, see Constructive combinatorics of Dickson's Lemma by Iosif Petrakis for several fine quantitative results). And even more: There is a procedure for regarding this proof—and many others in the theory of well quasi-orders—as *blueprints* for more informative constructive proofs. This shall be our motto for today:

Do not take classical proofs literally, instead ask which constructive proofs they are blueprints for.

The displayed stability results, along with several others, provide a flexible toolbox for constructing new well quasi-orders from given ones. However, with the classical formulation of *well*, shortly to be renamed "well$_\infty$", these results are inherently classical (even the weaker claim "the product of two streamless sets is streamless" admits countermodels).

(In Higman's lemma, the set X^* of finite lists of elements of X is equipped with the following ordering: We have $(x_0 :: \ldots :: x_{n-1} :: []) \leq (y_0 :: \ldots :: y_{m-1} :: [])$ iff there is an increasing injection $f : \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$ such that $x_i \leq y_{f(i)}$ for all $i < n$. Kruskal's theorem is the result behind the celebrated enormous number TREE(3).)

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well$_\infty$** iff every sequence $\mathbb{N} \to X$ is good.

Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well$_\infty$.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

offensive?

Key stability results

Assuming LEM and DC, …

Dickson: If X and Y are well$_\infty$, so is $X \times Y$.

Higman: If X is well$_\infty$, so is List X.

Kruskal: If X is well$_\infty$, so is Tree X.

$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$

$\leq$

2025-06-20

Towards topological type theory for decrypting transfinite methods in classical mathematics

└─A case study in Hilbert's program

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well$_\infty$** iff every sequence $\mathbb{N} \to X$ is good.

Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well$_\infty$.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. $\square$

offensive?

Key stability results

Assuming LEM and DC, …

Dickson: If X and Y are well$_\infty$, so is $X \times Y$.

Higman: If X is well$_\infty$, so is List X.

Kruskal: If X is well$_\infty$, so is Tree X.

$7,\quad 4,\quad 3,\quad 1,\quad 8,\quad 2,\quad \ldots$

$\leq$

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well$_\infty$** iff every sequence $\mathbb{N} \to X$ is good.

Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well$_\infty$.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. $\square$

offensive?

$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$

$\leq$

Key stability results

Constructively, ...

Dickson: If X and Y are well$_{\text{ind}}$, so is $X \times Y$.

Higman: If X is well$_{\text{ind}}$, so is $\text{List}\, X$.

Kruskal: If X is well$_{\text{ind}}$, so is $\text{Tree}\, X$.

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well$_\infty$** iff every sequence $\mathbb{N} \to X$ is good.

Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well$_\infty$.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. $\square$

offensive?

Key stability results

Constructively, …

Dickson: If X and Y are well$_{\text{ind}}$, so is $X \times Y$.

Higman: If X is well$_{\text{ind}}$, so is List X.

Kruskal: If X is well$_{\text{ind}}$, so is Tree X.

Def. A quasi-order X is **well$_{\text{ind}}$** iff $G\,[\,]$, where G is the following inductively defined predicate on **finite lists**. (In presence of bar induction, well$_{\text{ind}}$ $\Leftrightarrow$ well$_\infty$.)

$$\frac{p : \mathsf{Good}\,\sigma}{\mathsf{now}\,p : G\,\sigma} \qquad \frac{f : (x : X) \to G\,(\sigma ::^r x)}{\mathsf{later}\,f : G\,\sigma}$$

Thanks to work by Thierry Coquand, Daniel Fridlender and Monika Seisenberger, a constructive substitute is available, the notion well$_{\text{ind}}$. In classical mathematics (where bar induction is available), this notion is equivalent to well$_\infty$.

The predicate Good of finite lists $\sigma = (x_0 :: \ldots :: x_{n-1} :: [\,])$ appearing in the definition of G expresses that some earlier term is at most some later term, i. e. that there merely exist $i < j$ such that $x_i \leq x_j$.

The operator $::^r$ is adding an element at the end.

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
- The quasi-order X is **well$_\infty$** iff every sequence $\mathbb{N} \to X$ is good.

Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well$_\infty$.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

offensive?

Key stability results

Constructively, …

Dickson: If X and Y are well$_{\text{ind}}$, so is $X \times Y$.

Higman: If X is well$_{\text{ind}}$, so is List X.

Kruskal: If X is well$_{\text{ind}}$, so is Tree X.

Def. A quasi-order X is **well$_{\text{ind}}$** iff $G\,[\,]$, where G is the following inductively defined predicate on **finite lists**. (In presence of bar induction, well$_{\text{ind}}$ $\Leftrightarrow$ well$_\infty$.)

$$\frac{p : \mathrm{Good}\,\sigma}{\mathrm{now}\, p : G\,\sigma} \qquad \frac{f : (x : X) \to G\,(\sigma ::^r x)}{\mathrm{later}\, f : G\,\sigma}$$

Is there a procedure for reinterpreting **classical proofs** regarding well$_\infty$ as **blueprints for constructive proofs** regarding well$_{\text{ind}}$?

The original notion well$_\infty$:

- ✓ short and simple
- ✓ constructively satisfied for the main examples (but only because of the theory around well$_{\text{ind}}$)
- ✓ concise abstract proofs (albeit employing transfinite methods)
- ✗ main results not constructively attainable
- ✗ philosophically strenuous by the quantification over all sequences
- ✗ not stable under "change of base"—a forcing extension of the universe may well contain more sequences than the base universe
- ✗ negative (universal) condition

The constructive substitute well$_{\text{ind}}$:

- ✓ main results constructive
- ✓ stable under change of base
- ✓ positive (existential) condition
- ✗ proofs intriguing, but also somewhat alien, not just some trivial reshuffling of the classical arguments; classical sequence language cannot be used

Missing functions in the type of all functions?

Behold: A transfinite tool …

Lemma. LEM Let X be well$_\infty$. Let $\alpha : \mathbb{N} \to X$. Then there is an increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$.

Proof.

1. The type $I := \sum_{i\,:\,\mathbb{N}} \neg \sum_{j\,:\,\mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j$ is not in bijection with $\mathbb{N}$, as else the I-extracted subsequence of α would not be good.
2. By LEM, the type I is finite.
3. Any index i_0 larger than all the indices in I is a suitable starting point for an increasing subsequence. □

… implying a concrete consequence

Cor. LEM Let X and Y be well$_\infty$. Then $X \times Y$ is well$_\infty$.

Proof.

1. Let a sequence $\gamma : \mathbb{N} \to X \times Y$ be given. Write $\gamma\, k = (\alpha\, k, \beta\, k)$.
2. By the lemma, there is an increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$.
3. Because Y is well, there are indices $n < m$ such that $\beta\, i_n \leq \beta\, i_m$.
4. As also $\alpha\, i_n \leq \alpha\, i_m$, the sequence γ is good. □

We cannot trust LEM*-provided sequences to be available in the type* $\mathbb{N} \to X$*. Similarly with* DC.

The double negation translation technique fails to constructivize the two proofs presented on the slide.

The reason is that, after applying the translation, the "I-extracted subsequence" appearing in the proof on the left will no longer actually be a function $\mathbb{N} \to X$. Instead, it will be a partially defined function with the property that each number n does *not not* belong to the domain. However, the assumption that X is well$_\infty$ does not apply to these kinds of generalized sequences.

In contrast, the notion well$_{\text{ind}}$ is much stronger. If X is well$_{\text{ind}}$, then not only will every actual sequence $\mathbb{N} \to X$ be good, but (in a suitable sense) so will every up-to-$\neg\neg$-partially-defined sequence, every multivalued sequence and many more kinds of generalized sequences.

Where do many cherished inductive definitions come from?

In mathematics, we routinely enlarge structures:

- Pass from $\mathbb{Q}$ to $\mathbb{R}$, to embrace irrationals.
- Pass from $\mathbb{R}$ to $\mathbb{C}$, to obtain $\sqrt{-1}$.
- Pass from $\mathbb{C}$ to $\mathbb{C}[X]$, to obtain a "generic number".

In set and type theory, we can also enlarge **the mathematical universe**:

- Force a **generic sequence** $\mathbb{N} \to X$.
- Force a **generic enumeration** $\mathbb{N} \twoheadrightarrow X$ (even if X is uncountable).
- Force a **generic prime ideal** of a given ring.

The technique for enlarging the mathematical universe is called *forcing* and was originally pioneered by Paul Cohen. Set theorists use forcing to construct new models of set theory from given ones, in order to explore the range of set-theoretic possibility. For instance, by forcing we can construct models of ZFC validating the continuum hypothesis and also models which falsify it.

We here refer to a simplification of original forcing which is useful in a constructive metatheory. At its core, every forcing extension is just a formula and proof translation of a certain form. For instance, there is a forcing extension validating LEM even if the base universe does not; this forcing extension is not a deep mystery, for a statement holds in that forcing extension iff its double negation translation holds in the base universe and it is well-known that the double negation translation of every instance of LEM is an intuitionistic tautology.

Here is a set of slides on constructive forcing, and Section 4 of this joint paper with Peter Schuster contains a written summary of constructive forcing.

Where do many cherished inductive definitions come from?

In mathematics, we routinely enlarge structures:

- Pass from $\mathbb{Q}$ to $\mathbb{R}$, to embrace irrationals.
- Pass from $\mathbb{R}$ to $\mathbb{C}$, to obtain $\sqrt{-1}$.
- Pass from $\mathbb{C}$ to $\mathbb{C}[X]$, to obtain a "generic number".

In set and type theory, we can also enlarge **the mathematical universe**:

- Force a **generic sequence** $\mathbb{N} \to X$.
- Force a **generic enumeration** $\mathbb{N} \twoheadrightarrow X$ (even if X is uncountable).
- Force a **generic prime ideal** of a given ring.

Central observations of the multiversal yoga:

- A quasi-order X is well_{ind} iff the generic sequence $\mathbb{N} \to X$ is good.
- A set is Noetherian iff the generic sequence $\mathbb{N} \to X$ has two repeated terms.
- A relation is well-founded iff for the generic descending chain, $\perp$.
- A ring element is nilpotent iff it is contained in the generic prime ideal.

Forcing the existence of a new function $\mathbb{N} \to X$ is also called *Cohen forcing*. Forcing the existence of a new surjection is called *Lévy collapse*. Forcing the existence of a prime ideal does not have an established name, perhaps it should be called *Zariski forcing*; it has been explored by Thierry Coquand and others of the constructive algebra crew.

The multiversal yoga allows us to reason with the good notion well_{ind} using the simple language of well_{∞}, and similarly with several other notions.

Where do many cherished inductive definitions come from?

In mathematics, we routinely enlarge structures:

- Pass from $\mathbb{Q}$ to $\mathbb{R}$, to embrace irrationals.
- Pass from $\mathbb{R}$ to $\mathbb{C}$, to obtain $\sqrt{-1}$.
- Pass from $\mathbb{C}$ to $\mathbb{C}[X]$, to obtain a "generic number".

In set and type theory, we can also enlarge **the mathematical universe**:

- Force a **generic sequence** $\mathbb{N} \to X$.
- Force a **generic enumeration** $\mathbb{N} \twoheadrightarrow X$ (even if X is uncountable).
- Force a **generic prime ideal** of a given ring.

Central observations of the multiversal yoga:

- A quasi-order X is well_{ind} iff the generic sequence $\mathbb{N} \to X$ is good.
- A set is Noetherian iff the generic sequence $\mathbb{N} \to X$ has two repeated terms.
- A relation is well-founded iff for the generic descending chain, $\perp$.
- A ring element is nilpotent iff it is contained in the generic prime ideal.

The mystery of nongeometric sequents:

- The generic ring is a field.
- For the generic surjection $f : \mathbb{N} \twoheadrightarrow \mathbb{R}$, $\neg\neg\exists n : \mathbb{N}.\, f(n) = \pi \wedge f(n+1) = e$.

For more on the mystery of nongeometric sequents, see this set of slides.

A modal language for harnessing the multiverse

Def. A statement φ holds ...

- **everywhere** $(\Box\varphi)$ iff it holds **in every topos** (over the current base).
- **somewhere** $(\Diamond\varphi)$ iff it holds **in some positive topos**.
- **proximally** $(\diamondsuit\varphi)$ iff it holds **in some positive overt topos**.

We then have:

1. $\mathrm{Well}_{\mathrm{ind}}(X, \leq) \iff \Box\,\mathrm{Well}_{\infty}(X, \leq)$.
2. $(\Diamond\varphi) \iff \varphi$, if φ is a geometric implication ("$\forall \ldots \forall(\% \Rightarrow \%)$", with no $\forall$ nor $\Rightarrow$ in $\%$).
3. $(\diamondsuit\varphi) \iff \varphi$, if φ is a bounded first-order statement.
4. For every inhabited type X, $\diamondsuit\Box\,\mathrm{Countable}(X)$, where $\mathrm{Countable}(X) :\equiv (\exists f : \mathbb{N} \to X.\, \mathrm{Surjective}(f))$. So being countable is a **button**.
5. $\Diamond$ LEM. (Baby Barr / Friedman's trick / nontrivial exit continuation) In fact, LEM is a **switch**: $\Box((\Diamond\,\text{LEM}) \wedge (\Diamond\neg\,\text{LEM}))$
6. ZORN $\Rightarrow \Box\Diamond$ AC. (Great Barr)

By *topos*, we mean *Grothendieck topos*. In constructive forcing, a "forcing extension of the base universe" is exactly the same thing as a Grothendieck topos.

A particular member of the rich and varied landscape of toposes is the *trivial topos*, in which every statement whatsoever holds. By restricting to positive toposes, we exclude this special case.

For positive toposes $\mathcal{E}$, a geometric implication holds in $\mathcal{E}$ iff it holds in the base universe. For positive overt toposes $\mathcal{E}$, we even have that a bounded first-order formula holds in $\mathcal{E}$ iff it holds in the base. Hence, for the purpose of verifying a bounded first-order assertion about the base, we can freely pass to a positive overt topos with problem-adapted better higher-order properties (such as that some uncountable set from the base now appears countable, or that an infinite sequence whose existence is predicted by failing dependent choice now actually exists).

Here is a rough early draft of a preprint with more details about the modal multiverse.

A modal language for harnessing the multiverse

Def. A statement φ holds …

- **everywhere** $(\Box\varphi)$ iff it holds **in every topos** (over the current base).
- **somewhere** $(\Diamond\varphi)$ iff it holds **in some positive topos**.
- **proximally** $(\diamondsuit\varphi)$ iff it holds **in some positive overt topos**.

We then have:

1. $\mathsf{Well}_{\mathrm{ind}}(X, \leq) \iff \Box\mathsf{Well}_{\infty}(X, \leq)$.
2. $(\Diamond\varphi) \iff \varphi$, if φ is a geometric implication ("$\forall \ldots \forall(\% \Rightarrow \%)$", with no $\forall$ nor $\Rightarrow$ in $\%$).
3. $(\diamondsuit\varphi) \iff \varphi$, if φ is a bounded first-order statement.
4. For every inhabited type X, $\diamondsuit\Box\mathsf{Countable}(X)$,
   where $\mathsf{Countable}(X) :\equiv (\exists f : \mathbb{N} \to X.\, \mathsf{Surjective}(f))$.
   So being countable is a **button**.
5. $\Diamond$ LEM. (Baby Barr / Friedman's trick / nontrivial exit continuation)
   In fact, LEM is a **switch**: $\Box((\Diamond\,\text{LEM}) \wedge (\Diamond\neg\,\text{LEM}))$
6. ZORN $\Rightarrow \Box\Diamond$ AC. (Great Barr)

The idea to study the modal multiverse of toposes in a principled manner was proposed by Alexander Oldenziel in 2016. *Foreshadowed by:*

1984 André Joyal, Miles Tierney. "An extension of the Galois theory of Grothendieck".

1987 Andreas Blass. "Well-ordering and induction in intuitionistic logic and topoi".

2010s Milly Maietti, Steve Vickers. Ongoing work on arithmetic universes.

2011 Joel David Hamkins. "The set-theoretic multiverse".

2013 Shawn Henry. "Classifying topoi and preservation of higher order logic by geometric morphisms".

A modal language for harnessing the multiverse

Def. A statement φ holds ...

- **everywhere** ($\Box\varphi$) iff it holds **in every topos** (over the current base).
- **somewhere** ($\Diamond\varphi$) iff it holds **in some positive topos**.
- **proximally** ($\diamondsuit\!\!\!\!\diamond\,\varphi$) iff it holds **in some positive overt topos**.

We then have:

1. $\mathsf{Well}_{\mathrm{ind}}(X, \leq) \iff \Box\mathsf{Well}_{\infty}(X, \leq)$.
2. $(\Diamond\varphi) \iff \varphi$, if φ is a geometric implication ("$\forall \ldots \forall(\% \Rightarrow \%)$", with no $\forall$ nor $\Rightarrow$ in $\%$).
3. $(\diamondsuit\!\!\!\!\diamond\,\varphi) \iff \varphi$, if φ is a bounded first-order statement.
4. For every inhabited type X, $\diamondsuit\!\!\!\!\diamond\,\Box\mathsf{Countable}(X)$, where $\mathsf{Countable}(X) :\equiv (\exists f : \mathbb{N} \to X.\, \mathsf{Surjective}(f))$. So being countable is a **button**.
5. $\Diamond$ LEM. (Baby Barr / Friedman's trick / nontrivial exit continuation) In fact, LEM is a **switch**: $\Box\big((\Diamond \text{LEM}) \wedge (\Diamond\neg \text{LEM})\big)$
6. ZORN $\Rightarrow \Box\Diamond$ AC. (Great Barr)

With the modal language we seek to provide an accessible and modular framework for constructivization results.

For instance, conservativity of classical logic over intuitionistic logic for geometric implications (known under various names such as Barr's theorem, Friedman's trick, escaping the continuation monad, ...) is packaged up by the observation that *somewhere*, the law of excluded middle holds.

Another example: In the community around Krull's lemma, it is well-known that we can constructively infer that a given ring element $x \in A$ is nilpotent from knowing that it is contained in the *generic prime ideal* of A. This entity is not actually an honest prime ideal of the ring A in the base universe, but a certain combinatorial notion (efficiently dealt with using *entailment relations*). Constructive forcing allows us to reify the generic prime ideal as an actual prime ideal in a suitable forcing extension, so in a suitable topos (a version of the little Zariski topos of the ring).

1. Realize a generic gadget as some kind of limit of **approximations** from the base universe. For the generic function $f_0 : \mathbb{N} \to X$:

   L : Set
   L = List X

The plain combinatorics of toposes

1. Realize a generic gadget as some kind of limit of **approximations** from the base universe. For the generic function $f_0 : \mathbb{N} \to X$:

   L : Set
   L = List X

2. **Reinterpret**, in a mechanical fashion, assertions about the generic gadget as assertions about its approximations ( ).
   - We have the **stage-dependent proposition** "$f_0\ n = x$", a certain function $L \to \mathrm{Prop}$: $\lambda\sigma.\ (\mathrm{lookupMaybe}\,\sigma\ n = \mathrm{just}\ x)$

The plain combinatorics of toposes

1. Realize a generic gadget as some kind of limit of **approximations** from the base universe. For the generic function $f_0 : \mathbb{N} \to X$:

   ```
   L : Set
   L = List X
   ```

2. **Reinterpret**, in a mechanical fashion, assertions about the generic gadget as assertions about its approximations (⚙).
   - We have the **stage-dependent proposition** "$f_0\ n = x$", a certain function $L \to \mathrm{Prop}$: $\lambda\sigma.\ (\mathrm{lookupMaybe}\,\sigma\ n = \mathrm{just}\ x)$

3. Be prepared to **evolve approximations** (⚙).

   ```
   data ∇ (P : L → Prop) : L → Prop where
     now : {σ : L} → P σ → ∇ P σ
     later : {σ : L} → ((x : X) → ∇ P (σ ::ʳ x)) → ∇ P σ
   ```

   - For a stage-dependent proposition $P : L \to \mathrm{Prop}$, $\nabla\, P\, \sigma$ expresses that no matter how σ will evolve over a time to a better approximation τ, eventually $P\,\tau$ will hold.
   - That f_0 is defined on input n can be expressed as $\nabla\, P\, [\,]$ where $P\,\sigma :\equiv (\mathrm{length}\,\sigma > n)$.

Proofs (about the base universe) are (pure) programs; proofs about a forcing extension are effectful programs unfolding in the ∇ monad.

The ∇ monad is a refinement of the (monotonic) state monad.

The plain combinatorics of toposes

1. Realize a generic gadget as some kind of limit of **approximations** from the base universe. For the generic function $f_0 : \mathbb{N} \to X$:

   L : Set
   L = List X

2. **Reinterpret**, in a mechanical fashion, assertions about the generic gadget as assertions about its approximations (⚙).
   - We have the **stage-dependent proposition** "$f_0\ n = x$", a certain function $L \to \mathrm{Prop}$: $\lambda\sigma.\ (\mathrm{lookupMaybe}\,\sigma\ n = \mathrm{just}\ x)$

3. Be prepared to **evolve approximations** (⚙).

   data ∇ $(P : \mathsf{L} \to \mathsf{Prop}) : \mathsf{L} \to \mathsf{Prop}$ where
     now : $\{\sigma : \mathsf{L}\} \to P\ \sigma \to \nabla\ P\ \sigma$
     later : $\{\sigma : \mathsf{L}\} \to ((x : \mathsf{X}) \to \nabla\ P\ (\sigma ::^r x)) \to \nabla\ P\ \sigma$

   - For a stage-dependent proposition $P : L \to \mathrm{Prop}$, $\nabla\, P\, \sigma$ expresses that no matter how σ will evolve over a time to a better approximation τ, eventually $P\,\tau$ will hold.
   - That f_0 is defined on input n can be expressed as $\nabla\, P\,[\,]$ where $P\,\sigma :\equiv (\mathrm{length}\,\sigma > n)$.

4. Crucially, this interpretation is **sound** with respect to constructive reasoning.

Proofs (about the base universe) are (pure) programs; proofs about a forcing extension are effectful programs unfolding in the ∇ monad.

The ∇ monad is a refinement of the (monotonic) state monad.

Formal metatheory

✓ *There are type-theoretic multiverses*, such as

- the collection of all $\mathrm{PSh}(\mathcal{C} \times \mathcal{B})$, where $\mathcal{B}$ ranges over cube categories and $\mathcal{C}$ over arbitrary small categories, and their corresponding sheaf models

  Coquand. "A survey of constructive presheaf models of univalence". *ACM SIGLOG News*, 5.3 (2018).

Formal metatheory

✓ *There are type-theoretic multiverses,* such as

- the collection of all $\mathrm{PSh}(\mathcal{C} \times \mathcal{B})$, where $\mathcal{B}$ ranges over cube categories and $\mathcal{C}$ over arbitrary small categories, and their corresponding sheaf models

  Coquand. "A survey of constructive presheaf models of univalence". *ACM SIGLOG News*, 5.3 (2018).

✗ *Accessing the multiverse from within intensional type theory is tricky:*

- Given a model of $\mathfrak{s}$CIC and a category $\mathcal{C}$ in it, we have a syntactic presheaf model of CIC.

  Coquand, Jaber. "A note on forcing and type theory". *Fundamenta Informaticae 100* (2010).
  Jaber, Lewertowski, Pédrot, Sozeau, Tabareau. "The definitional side of the forcing". *Proceedings of LICS '16* (2016).
  Pédrot. "Russian constructivism in a prefascist theory". *Proceedings of LICS '20* (2020).

- Given a suitable lex modality, we have a syntactic sheaf model (model of modal types).

  Coquand, Ruch, Sattler. "Constructive sheaf models of type theory." *Math. Struct. Comput. Sci.* 31.9 (2021).
  Escardó, Xu. "Sheaf models of type theory in type theory". Unpublished (2016).
  Quirin. "Lawvere–Tierney sheafification in Homotopy Type Theory". PhD thesis (2016).

- (I believe) we have syntactic sheaf models in certain special cases, when no coherence issues arise in defining the notion of presheaves.

Note: We can use ∇ even without a proper metatheoretic backing.

Exciting ongoing work on split type theory by Martin Baillon, Assia Mahboubi and Pierre-Marie Pédrot!

Increasing subsequences as convenient fictions

Let X be a quasi-order. Let $B : \mathsf{List}\, X \to \mathsf{Prop}$ be a monotone predicate.

Classical blueprint

Thm. LEM If X is well$_\infty$ and if every increasing sequence $\alpha : \mathbb{N} \to X$ has a prefix validating B, then every sequence has a prefix validating B.

Proof. Let $\alpha : \mathbb{N} \to X$ be a sequence. By the lemma, there is an increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$. By assumption, this subsequence has a prefix validating B. This prefix is part of a prefix of the original sequence α. Hence we conclude by monotonicity. □

Constructive reimagination

Thm. If X is well$_{\mathrm{ind}}$ and if $\nabla^{\nearrow} B\,[\,]$, then $\nabla B\,[\,]$. ⚙

Proof. Let $\alpha : \mathbb{N} \to X$ be a sequence (in an arbitrary topos). *Somewhere,* LEM holds. *There* X is still well$_\infty$, so that we have an increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$. By assumption, this subsequence has a finite prefix validating B. This prefix is part of a prefix of the original sequence α. Hence α has a prefix validating B by monotonicity *there*. So *somewhere* there is a finite prefix validating B. Thus there actually is a finite prefix validating B. □

On this slide ∇ is the monad described on slide 5, and $\nabla^{\nearrow}$ is the variant where lists are only allowed to grow in a monotonically increasing manner; so ∇ is the monad for the generic sequence $\mathbb{N} \to X$, while $\nabla^{\nearrow}$ is the monad for the generic increasing sequence $\mathbb{N} \to X$.

The displayed multiversal proof gives a positive answer to a question posed by Stefano Berardi, Gabriele Buriola and Peter Schuster, see this set of slides.

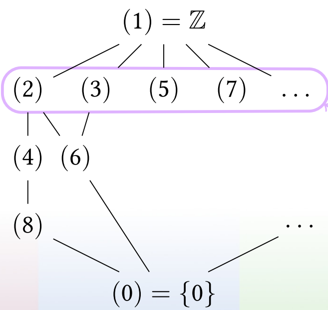
At the end of the day, this proof yields a concrete algorithm (see here for manually-extracted Perl code). Given an infinite sequence, this algorithm seeks, in a certain systematic but not exhaustive manner, a sufficiently long monotonic subsequence, making judicious use of backtracking. This search terminates, though this fact is not obvious just by looking at the algorithm.

Maximal ideals as convenient fictions

Let A be a commutative ring with unit.

Thm. Let M be a surjective matrix with more rows than columns over A. Then $1 = 0$ in A.

Classical proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix M is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. □



maximal among the proper ideals

- $\neg(1 \in \mathfrak{m})$
- $\neg\big(1 \in \mathfrak{m} + (x)\big) \Rightarrow x \in \mathfrak{m}$

The displayed classical proof is quite efficient from the point of view of organizing mathematical knowledge, as it quickly reduces the general situation of dealing with an arbitrary ring to dealing with a field. Alas, read literally, it is hopeless ineffective.

Maximal ideals as convenient fictions

Let A be a commutative ring with unit.

Thm. Let M be a surjective matrix with more rows than columns over A. Then $1 = 0$ in A.

Classical proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix M is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. $\square$

Multiversal constructive proof. We may work *somewhere* where LEM holds. So assume not. *Proximally*, there is a maximal ideal $\mathfrak{m}$ (⚙). The matrix M is still surjective *there*, and also over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. $\square$

The displayed classical proof is quite efficient from the point of view of organizing mathematical knowledge, as it quickly reduces the general situation of dealing with an arbitrary ring to dealing with a field. Alas, read literally, it is hopeless ineffective.

By employing modal language, we can closely mimic the original proof and be fully constructive at the same time.

Maximal ideals as convenient fictions

Let A be a commutative ring with unit.

Thm. Let M be a surjective matrix with more rows than columns over A. Then $1 = 0$ in A.

Classical proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix M is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. $\square$

Multiversal constructive proof. We may work *somewhere* where LEM holds. So assume not. *Proximally,* there is a maximal ideal $\mathfrak{m}$ (⚙). The matrix M is still surjective *there*, and also over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. $\square$

Unrolled constructive proof (special case, ⚙). Write $M = \binom{x}{y}$. By surjectivity, have u, v with

$$u\binom{x}{y} = \binom{1}{0} \quad \text{and} \quad v\binom{x}{y} = \binom{0}{1}.$$

Hence $1 = (vy)(ux) = (uy)(vx) = 0$. $\square$

The displayed classical proof is quite efficient from the point of view of organizing mathematical knowledge, as it quickly reduces the general situation of dealing with an arbitrary ring to dealing with a field. Alas, read literally, it is hopeless ineffective.

By employing modal language, we can closely mimic the original proof and be fully constructive at the same time.

By unwinding all modal definitions, the modal proof can be unrolled to a fully explicit computation.

A commutative ring A with unit is ...

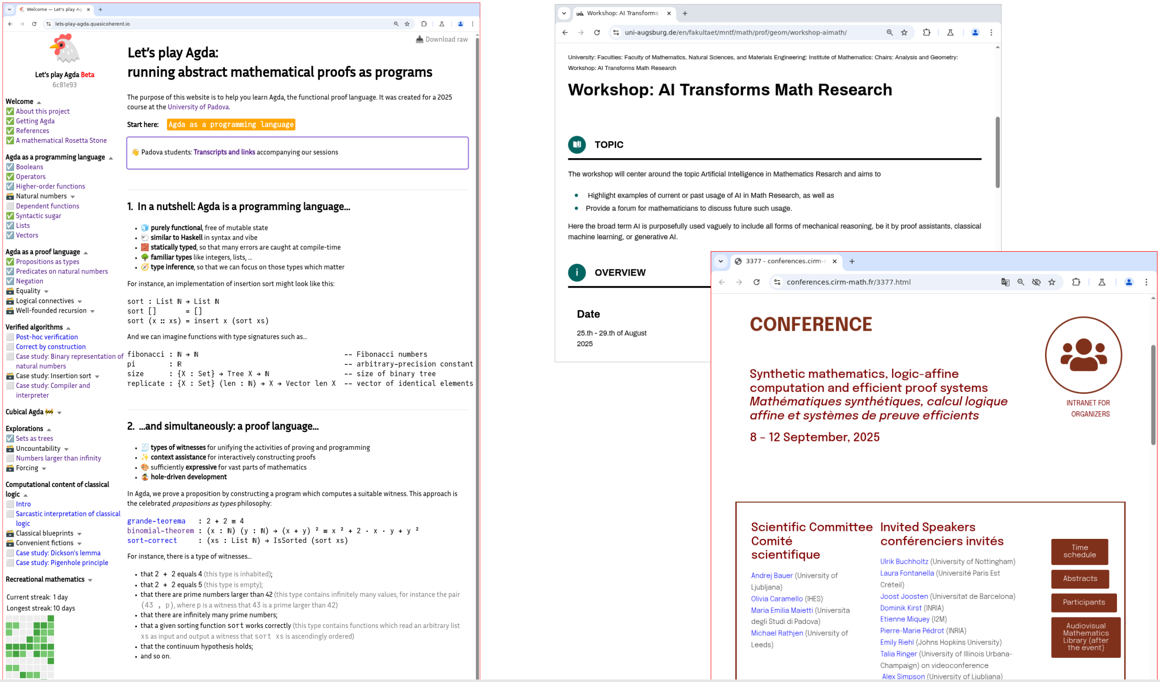
- Noetherian$_\infty$ iff for every sequence $x_0, x_1, \ldots$ of ring elements, there is a number $n : \mathbb{N}$ such that $x_n, x_{n+1}, x_{n+2}, \ldots \in (x_0, \ldots, x_{n-1})$.
- Noetherian$_{\mathrm{RS}}$ iff for every sequence $x_0, x_1, \ldots$ of ring elements, there is a number $n : \mathbb{N}$ such that $x_n \in (x_0, \ldots, x_{n-1})$.
- Noetherian$_{\mathrm{ind}}$ iff a certain inductively defined condition holds [Coquand–Persson].

We then have:

$$\begin{aligned} \mathsf{Noetherian}_{\mathrm{ind}}(A) &\Longleftrightarrow \Box\,\mathsf{Noetherian}_{\mathrm{RS}}(A) \\ &\Longleftrightarrow \Box\big(\mathrm{LEM} \Rightarrow \mathsf{Noetherian}_\infty(A)\big) \\ &\Longleftrightarrow \Box\Diamond\,\mathsf{Noetherian}_\infty(A). \end{aligned}$$

With this observation, from their classical counterparts, we can extract constructive proofs of:

1. Hilbert's basis theorem: If A is Noetherian$_{\mathrm{ind}}$, then so is $A[X]$.
2. Locality: If $1 = f_1 + \ldots + f_n$ and if each ring $A[f_i^{-1}]$ is Noetherian$_{\mathrm{ind}}$, then so is A. (As a consequence, A is Noetherian$_{\mathrm{ind}}$ iff algebraic geometry's $\mathcal{O}_{\mathrm{Spec}(A)}$ is.)



2025-06-20

Towards topological type theory for decrypting transfinite methods in classical mathematics

Let's play Agda, an interactive course on Agda, complementing the Agdapad.

AI Transforms Maths Research, a workshop in Augsburg.

Synthetic mathematics, a conference in Luminy.

Backup slides.

Ingredients for forcing

To construct a forcing extension, we require:

1. a base universe V
2. a preorder L of **forcing conditions** in V, pictured as **finite approximations** (*convention:* $\tau \preccurlyeq \sigma$ means that τ is a better finite approximation than σ)
3. a **covering system** governing how finite approximations evolve to better ones (for each $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$, with a simulation condition)

In the forcing extension V^∇, there will then be a **generic filter** (ideal object).

Ingredients for forcing

To construct a forcing extension, we require:

1. a base universe V
2. a preorder L of **forcing conditions** in V, pictured as **finite approximations** (*convention:* $\tau \preccurlyeq \sigma$ means that τ is a better finite approximation than σ)
3. a **covering system** governing how finite approximations evolve to better ones (for each $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$, with a simulation condition)

In the forcing extension V^∇, there will then be a **generic filter** (ideal object).

For the generic surjection $\mathbb{N} \twoheadrightarrow X$

Use **finite lists** $\sigma \in X^*$ as forcing conditions, where $\tau \preccurlyeq \sigma$ iff σ is an initial segment of τ, and be prepared to grow σ to …

(a) one of $\{\sigma x \mid x \in X\}$, to make σ more defined

(b) one of $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$, for any $a \in X$, to make σ more surjective

Ingredients for forcing

To construct a forcing extension, we require:

1. a base universe V
2. a preorder L of **forcing conditions** in V, pictured as **finite approximations** (*convention:* $\tau \preccurlyeq \sigma$ means that τ is a better finite approximation than σ)
3. a **covering system** governing how finite approximations evolve to better ones (for each $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$, with a simulation condition)

In the forcing extension V^∇, there will then be a **generic filter** (ideal object).

For the generic surjection $\mathbb{N} \twoheadrightarrow X$

Use **finite lists** $\sigma \in X^*$ as forcing conditions, where $\tau \preccurlyeq \sigma$ iff σ is an initial segment of τ, and be prepared to grow σ to …

(a) one of $\{\sigma x \mid x \in X\}$, to make σ more defined

(b) one of $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$, for any $a \in X$, to make σ more surjective

For the generic prime ideal of a ring A

Use **f.g. ideals** as forcing conditions, where $\mathfrak{b} \preccurlyeq \mathfrak{a}$ iff $\mathfrak{b} \supseteq \mathfrak{a}$, and be prepared to grow $\mathfrak{a}$ to …

(a) one of $\emptyset$, if $1 \in \mathfrak{a}$, to make $\mathfrak{a}$ more proper

(b) one of $\{\mathfrak{a} + (x), \mathfrak{a} + (y)\}$, if $xy \in \mathfrak{a}$, to make $\mathfrak{a}$ more prime

The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

- Repeats $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i = x_j$
- Good $\quad x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i \leq x_j \quad$ (for some preorder $\leq$ on X)

The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

- Repeats $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i = x_j$
- Good $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i \leq x_j$ (for some preorder $\leq$ on X)

We then define "$\nabla\, P\, \sigma$" ("P bars σ") inductively by the following clauses:

1. If $P\sigma$, then $\nabla\, P\, \sigma$.
2. If $\nabla\, P\, \tau$ for all $\tau \in R$, where R is some covering of σ, then $\nabla\, P\, \sigma$.

So $\nabla\, P\, \sigma$ expresses in a **direct inductive fashion**:

> "No matter how σ evolves to a better approximation τ, eventually $P\, \tau$ will hold."

The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

- Repeats $x_0 \ldots x_{n-1} :\equiv \exists i. \exists j. i < j \wedge x_i = x_j$
- Good $x_0 \ldots x_{n-1} :\equiv \exists i. \exists j. i < j \wedge x_i \leq x_j$ (for some preorder $\leq$ on X)

We then define "$\nabla P \sigma$" ("P bars σ") inductively by the following clauses:

1. If $P\sigma$, then $\nabla P \sigma$.
2. If $\nabla P \tau$ for all $\tau \in R$, where R is some covering of σ, then $\nabla P \sigma$.

So $\nabla P \sigma$ expresses in a **direct inductive fashion**:

> "No matter how σ evolves to a better approximation τ, eventually $P\tau$ will hold."

We use quantifier-like notation: "$\nabla(\tau \preccurlyeq \sigma). P\tau$" means "$\nabla P \sigma$".

Thm. Every IQC-proof remains correct, with at most a polynomial increase in length, if throughout we replace

$$\begin{array}{rcll} \exists & \rightsquigarrow & \exists^{\mathrm{cl}}, \quad \text{where} & \exists^{\mathrm{cl}} :\equiv \neg\neg\exists, \\ \vee & \rightsquigarrow & \vee^{\mathrm{cl}}, \quad \text{where} & \alpha \vee^{\mathrm{cl}} \beta :\equiv \neg\neg(\alpha \vee \beta), \\ = & \rightsquigarrow & =^{\mathrm{cl}}, \quad \text{where} & s =^{\mathrm{cl}} t :\equiv \neg\neg(s = t). \end{array}$$

Proof translations

Thm. Every IQC-proof remains correct, with at most a polynomial increase in length, if throughout we replace

$$\begin{array}{ccclrcl} \exists & \rightsquigarrow & \exists^{\mathrm{cl}}, & \text{where} & \exists^{\mathrm{cl}} & :\equiv & \neg\neg\exists, \\ \vee & \rightsquigarrow & \vee^{\mathrm{cl}}, & \text{where} & \alpha \vee^{\mathrm{cl}} \beta & :\equiv & \neg\neg(\alpha \vee \beta), \\ = & \rightsquigarrow & =^{\mathrm{cl}}, & \text{where} & s =^{\mathrm{cl}} t & :\equiv & \neg\neg(s = t). \end{array}$$

When we say: some statement "holds in $V^{\neg\neg}$",
we mean: its translation holds in V.

Similarly for arbitrary forcing extensions V^{∇}, "just with ∇ instead of $\neg\neg$".

Thm. Every IQC-proof remains correct, with at most a polynomial increase in length, if throughout we replace

$$\begin{array}{ccclrcl} \exists & \rightsquigarrow & \exists^{\mathrm{cl}}, & \text{where} & \exists^{\mathrm{cl}} & :\equiv & \neg\neg\exists, \\ \vee & \rightsquigarrow & \vee^{\mathrm{cl}}, & \text{where} & \alpha \vee^{\mathrm{cl}} \beta & :\equiv & \neg\neg(\alpha \vee \beta), \\ = & \rightsquigarrow & =^{\mathrm{cl}}, & \text{where} & s =^{\mathrm{cl}} t & :\equiv & \neg\neg(s = t). \end{array}$$

When we say: some statement "holds in $V^{\neg\neg}$",
we mean: its translation holds in V.

Similarly for arbitrary forcing extensions V^{∇}, "just with ∇ instead of $\neg\neg$".

Ex. As $\neg\neg(\varphi \vee \neg\varphi)$ is a theorem of IQC, the law of excluded middle holds in $V^{\neg\neg}$.

The ∇-translation

For bounded first-order formulas over the (large) first-order signature which has

1. one sort $\underline{X}$ for each set X in the base universe,
2. one n-ary function symbol $\underline{f} : \underline{X_1} \times \cdots \times \underline{X_n} \to \underline{Y}$ for each map $f : X_1 \times \cdots \times X_n \to Y$,
3. one n-ary relation symbol $\underline{R} \hookrightarrow \underline{X_1} \times \cdots \times \underline{X_n}$ for each relation $R \subseteq X_1 \times \cdots \times X_n$, and
4. an additional unary relation symbol $G \hookrightarrow \underline{L}$ (for the *generic filter* of L),

we recursively define:

$\sigma \models s = t$	iff	$\nabla\sigma.\, [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\, (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\, \sigma \preccurlyeq [\![\tau]\!]$.
$\sigma \models \top$	iff	$\top$.	$\sigma \models \bot$	iff	$\nabla\sigma.\, \bot$
$\sigma \models \varphi \wedge \psi$	iff	$(\sigma \models \varphi) \wedge (\sigma \models \psi)$.	$\sigma \models \varphi \vee \psi$	iff	$\nabla\sigma.\, (\sigma \models \varphi) \vee (\sigma \models \psi)$.
$\sigma \models \forall(x : \underline{X}).\, \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\, \forall(x_0 \in X).\, \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\, \varphi$	iff	$\nabla\sigma.\, \exists(x_0 \in X).\, \sigma \models \varphi[\underline{x_0}/x]$.

Finally, we say that φ "holds in V^∇" iff for all $\sigma \in L$, $\sigma \models \varphi$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma{\in}X^*).\, \forall(a{\in}X).\, \nabla(\tau{\preccurlyeq}\sigma).\, \exists(n{\in}\mathbb{N}).\, \tau[n] = a$.

The ∇-translation

$\sigma \models s = t$	iff	$\nabla\sigma.\, [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\, (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\, \sigma \preccurlyeq [\![\tau]\!]$.
$\sigma \models \top$	iff	$\top$.	$\sigma \models \bot$	iff	$\nabla\sigma.\, \bot$
$\sigma \models \varphi \wedge \psi$	iff	$(\sigma \models \varphi) \wedge (\sigma \models \psi)$.	$\sigma \models \varphi \vee \psi$	iff	$\nabla\sigma.\, (\sigma \models \varphi) \vee (\sigma \models \psi)$.
$\sigma \models \forall(x : \underline{X}).\, \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\, \forall(x_0 \in X).\, \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\, \varphi$	iff	$\nabla\sigma.\, \exists(x_0 \in X).\, \sigma \models \varphi[\underline{x_0}/x]$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma{\in}X^*).\, \forall(a{\in}X).\, \nabla(\tau{\preccurlyeq}\sigma).\, \exists(n{\in}\mathbb{N}).\, \tau[n] = a$.
map $\mathbb{N} \to X$	"the gen. sequence is good"	Good $\mid$ [ ].
frame of opens	"every complex number has a square root"	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.

The ∇-translation

$\sigma \models s = t$	iff	$\nabla\sigma.\, [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\, (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\, \sigma \preccurlyeq [\![\tau]\!]$.
$\sigma \models \top$	iff	$\top$.	$\sigma \models \bot$	iff	$\nabla\sigma.\, \bot$
$\sigma \models \varphi \wedge \psi$	iff	$(\sigma \models \varphi) \wedge (\sigma \models \psi)$.	$\sigma \models \varphi \vee \psi$	iff	$\nabla\sigma.\, (\sigma \models \varphi) \vee (\sigma \models \psi)$.
$\sigma \models \forall(x : \underline{X}).\, \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\, \forall(x_0 \in X).\, \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\, \varphi$	iff	$\nabla\sigma.\, \exists(x_0 \in X).\, \sigma \models \varphi[\underline{x_0}/x]$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma{\in}X^*).\, \forall(a{\in}X).\, \nabla(\tau{\preccurlyeq}\sigma).\, \exists(n{\in}\mathbb{N}).\, \tau[n] = a$.
map $\mathbb{N} \to X$	"the gen. sequence is good"	Good $\mid [\,]$.
frame of opens	"every complex number has a square root"	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.
little Zariski	"every f.g. vector space does *not not* have a basis"	**Grothendieck's generic freeness lemma**

Outlook

Passing to and from extensions

Thm. Let φ be a **bounded first-order formula** not mentioning G. In each of the following situations, we have that φ holds in V^∇ iff φ holds in V:

1. L and all coverings are inhabited (proximality).
2. L contains a top element, every covering of the top element is inhabited, and φ is a coherent implication (positivity).

The mystery of nongeometric sequents

The **generic ideal** of a ring is maximal:

$$(x \in \mathfrak{a} \Rightarrow 1 \in \mathfrak{a}) \Longrightarrow 1 \in \mathfrak{a} + (x).$$

The **generic ring** is a field:

$$(x = 0 \Rightarrow 1 = 0) \Longrightarrow (\exists y.\, xy = 1).$$

Traveling the multiverse …

LEM is a **switch** and **holds positively**; being countable is a **button**.

Every instance of DC **holds proximally**.

A geometric implication is provable iff it holds **everywhere**.

… upwards, but always keeping ties to the base.

Def. A **forcing notion** consists of a preorder L of **forcing conditions**, and for every $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$ of **coverings** of σ such that: If $\tau \preccurlyeq \sigma$ and $R \in \mathrm{Cov}(\sigma)$, there should be a covering $S \in \mathrm{Cov}(\tau)$ such that $S \subseteq \downarrow R$.

	preorder L	coverings of an element $\sigma \in L$	filters of L
1	X^*	$\{\sigma x \mid x \in X\}$	maps $\mathbb{N} \to X$
2	X^*	$\{\sigma x \mid x \in X\}$, $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$ for each $a \in X$	surjections $\mathbb{N} \twoheadrightarrow X$
3	f.g. ideals	—	ideals
4	f.g. ideals	$\{\sigma + (a), \sigma + (b)\}$ for each $ab \in \sigma$, $\{\}$ if $1 \in \sigma$	prime ideals
5	opens	$\mathcal{U}$ such that $\sigma = \bigcup \mathcal{U}$	points
6	$\{\star\}$	$\{\star \mid \varphi\} \cup \{\star \mid \neg\varphi\}$	witnesses of LEM

Def. A *filter* of a forcing notion (L, Cov) is a subset $F \subseteq L$ such that

1. F is upward-closed: if $\tau \preccurlyeq \sigma$ and if $\tau \in F$, then $\sigma \in F$;
2. F is downward-directed: F is inhabited, and if $\alpha, \beta \in F$, then there is a common refinement $\sigma \preccurlyeq \alpha, \beta$ such that $\sigma \in F$; and
3. F splits the covering system: if $\sigma \in F$ and $R \in \mathrm{Cov}(\sigma)$, then $\tau \in F$ for some $\tau \in R$.