An algebraic internal groupoid model of Martin-Löf type theory

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Comprehension category

Recall:

Definition ([Jac93])

A comprehension category is a strictly commutative diagram of functors

$$\begin{array}{ccc} F & \longrightarrow & C^{\mathbf{2}} \\ & \searrow \quad \swarrow_{\text{cod}} & \\ & C & \end{array}$$

such that $F \to C$ is a Grothendieck fibration and $F \to C^{\mathbf{2}}$ is a cartesian functor.

such data gives a (weak) model of MLTT.

Example: the groupoid model

An isofibration in **Gpd** is

$$\begin{array}{ccc} \mathscr{A} & & & a \\ \big\downarrow F & & & \big\downarrow \\ \mathscr{B} & & b \longrightarrow & F(a) \end{array}$$

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Example ([HS98])

$$\begin{array}{ccc} \mathbf{Isofib} & \hookrightarrow & \mathbf{Gpd}^2 \\ & {}_{\text{cod}}\searrow \quad \swarrow_{\text{cod}} & \\ & \mathbf{Gpd} & \end{array}$$

forms a comprehension category.

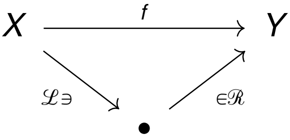
Weak Factorisation Systems

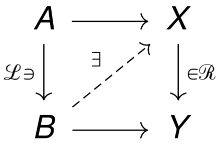
Many examples of comprehension categories arise from weak factorisation systems.

Definition

A *weak factorisation system* (wfs) on a category **C** is a pair $(\mathscr{L}, \mathscr{R})$ of classes of morphisms in **C** such that:

1. Every map $f : X \to Y$ can be factorised as a map in $\mathscr{L}$ followed by a map in $\mathscr{R}$.
2. $\mathscr{L} = {}^{\pitchfork}\mathscr{R}$ and $\mathscr{R} = \mathscr{L}^{\pitchfork}$.





Examples of WFSs

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(monomorphic homotopy equivalences, Kan fibrations) form a weak factorisation system on **sSet**.

... but can't often be equipped with coherent path objects. A fix of this was suggested by Garner in moving to the algebraic setting.

Type theoretic awfs

Definition ([GL23])

A *type-theoretic algebraic weak factorisation system* on a category **C** is a pair $(\mathbb{L}, \mathbb{R})$ of a comonad and a monad on $\mathbf{C}^{\rightarrow}$ such that $(\overline{\mathbb{L}\text{-Coalg}}, \overline{\mathbb{R}\text{-Alg}})$ is a wfs on **C** with some extra structure and satisfying certain conditions.

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Theorem ([GL23], Theorem 4.12)

Let $(\mathbb{C}, \mathbb{F})$ be a type theoretic algebraic weak factorisation system. Then the right adjoint splitting of the comprehension category associated to the awfs is equipped with strictly stable choices of Σ, Π and $\mathtt{Id}$-types i.e. it forms a model of MLTT.

The $\mathbb{F}$-algebras model the dependent types.

Cloven isofibrations

An isofibration in **Gpd** is

$$
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\mathscr{A} & \quad & \exists a' & \dashrightarrow & a \\
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i.e. the map $(F_1, d_0) : A_1 \to B_1 \times_{B_0} A_0$ has a section.

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i.e. the map $(F_1, d_0) : A_1 \to B_1 \times_{B_0} A_0$ has a section.

Definition

A cloven isofibration is a pair (F, s) in which the map $s : B_1 \times_{B_0} A_0 \to A_1$ is a chosen section of (F_1, d_0).

The algebraic groupoid model

Proposition

There is a monad $\mathbb{F} : \mathbf{Gpd}^{\mathbf{2}} \to \mathbf{Gpd}^{\mathbf{2}}$ such that $\mathbb{F}$-Alg $\cong$ **Cloven Isofibrations**.

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Proposition

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Theorem

There is a type theoretic AWFS involving $\mathbb{F}$. Hence cloven isofibrations model MLTT.

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Proposition

There is a monad $\mathbb{F} : \mathbf{Gpd}^2 \to \mathbf{Gpd}^2$ such that $\mathbb{F}$-Alg $\cong$ **CloveInIsofibrations**.

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There is a type theoretic AWFS involving $\mathbb{F}$. Hence cloven isofibrations model MLTT.

In the type theory, $(F, s) \neq (F, t)$ for $s \neq t$.

Internal groupoids

Definition

A *small groupoid* is:

$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \;\begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow[p_2]{} \end{array}\; C_1 \circlearrowleft^{(-)^{-1}} \;\begin{array}{c} \xrightarrow{d_1} \\ \xleftarrow{i} \\ \xrightarrow[d_0]{} \end{array}\; C_0$$

Where $C_0, C_1 \in \mathbf{Set}$. These are the objects of a $(2,1)$-category **Gpd**.

Internal groupoids

Definition

Let $\mathscr{E}$ be a category with pullbacks.
A groupoid internal to $\mathscr{E}$ is:

$$\cdots \longrightarrow C_1 \times_{C_0} C_1 \;\substack{\xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow[p_2]{}}\; C_1 \circlearrowleft_{(-)^{-1}} \;\substack{\xrightarrow{d_1} \\ \xleftarrow{i} \\ \xrightarrow[d_0]{}}\; C_0$$

Where $C_0, C_1 \in \mathscr{E}$.
These are the objects of a $(2,1)$-category $\mathbf{Gpd}(\mathscr{E})$.

Internal cloven isofibrations

Definition

An internal cloven isofibration is a pair (F, s) in which the map $s : B_1 \times_{B_0} A_0 \to A_1$ is a chosen section of $(F_1, d_0) : A_1 \to B_1 \times_{B_0} A_0$.

The internal algebraic groupoid model

Let $\mathscr{E}$ be a locally cartesian closed lextensive categories in which the forgetful functor $\mathscr{U}$: $\mathbf{Cat}(\mathscr{E}) \to \mathbf{Gph}(\mathscr{E})$ has a left adjoint:

The internal algebraic groupoid model

Let $\mathcal{E}$ be a locally cartesian closed lextensive categories in which the forgetful functor $\mathcal{U} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Gph}(\mathcal{E})$ has a left adjoint:

Theorem

There is a monad $\mathbb{F} : \mathbf{Gpd}(\mathcal{E})^{\mathbf{2}} \to \mathbf{Gpd}(\mathcal{E})^{\mathbf{2}}$ *such that* $\mathbb{F}$*-Alg* $\cong$ **CloveIsofibrations**. *Moreover, there is a type theoretic AWFS involving* $\mathbb{F}$*. Hence internal cloven isofibrations model MLTT.*

Non-examples

Non-examples: $\mathscr{E} = \mathbf{Cat}$, $\mathbf{Cat}(\mathscr{E})$, **Ab**...

Examples

Locally cartesian closed lextensive categories in which the forgetful functor $\mathscr{U} : \mathbf{Cat}(\mathscr{E}) \to \mathbf{Gph}(\mathscr{E})$ has a left adjoint:

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- Arithmetic Π-pretoposes [Mai10].
- Palmgren's CETCS [Pal12].
- **Asm** (cf. the effective topos [Hyl88])...

Future work

In joint work with Sam Speight:

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- Moreover, it forms a model of MLTT.
- We can find a modest discrete opfibration classifier in $\mathbf{Gpd}(\mathbf{Asm})$ (cf. [Web07]). In the type theory, this gives a univalent universe of modest 0-types.

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- Moreover, it forms a model of MLTT.
- We can find a modest discrete opfibration classifier in $\mathbf{Gpd}(\mathbf{Asm})$ (cf. [Web07]). In the type theory, this gives a univalent universe of modest 0-types.
- Moreover, we show that modest discrete opfibrations form a 2-category with a class of small discrete opfibrations (cf. [JM95]).

Future work II

Can we do this for $\mathbf{s}\mathscr{E} := [\Delta^{\mathrm{op}}, \mathscr{E}]$ and/or $[\square^{\mathrm{op}}, \mathscr{E}]$?

ArXiv link

The algebraic internal groupoid model of Martin-Löf type theory, 2025.

https://arxiv.org/abs/2503.17319

References I

John Bourke and Richard Garner.
Two-dimensional regularity and exactness.
Journal of Pure and Applied Algebra, 218(7):1346–1371, 2014.

Nicola Gambino and Marco Federico Larrea.
Models of martin-löf type theory from algebraic weak factorisation systems.
The Journal of Symbolic Logic, 88(1):242–289, 2023.

Martin Hofmann and Thomas Streicher.
The groupoid interpretation of type theory.
Twenty-five years of constructive type theory (Venice, 1995), 36:83–111, 1998.

References II

J.M.E. Hyland.
A small complete category.
Annals of Pure and Applied Logic, 40(2):135–165, 1988.

Bart Jacobs.
Comprehension categories and the semantics of type dependency.
Theoretical Computer Science, 107(2):169–207, 1993.

André Joyal and Ieke Moerdijk.
Algebraic set theory, volume 220.
Cambridge University Press, 1995.

References III

Maria Emilia Maietti.
Joyal's arithmetic universe as list-arithmetic pretopos.
Theory & Applications of Categories, 24, 2010.

Erik Palmgren.
Constructivist and structuralist foundations: Bishop's and Lawvere's theories of sets.
Annals of Pure and Applied Logic, 163(10):1384–1399, 2012.

Mark Weber.
Yoneda structures from 2-toposes.
Applied Categorical Structures, 15:259–323, 2007.