Choice principles and a cotopological modality in HoTT

Owen Milner
Department of Philosophy, Carnegie Mellon University

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1. Context

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They observed some of that material is suitable to be translated into HoTT.

This is done and there's now a formalization in Cubical Agda.

Amongst the results is a proof <u>in HoTT</u> that any of these forms of the axioms of countable choice imply the existence of an ∞-truncation modality.

2. What is ∞-truncation?

2.1. Connectivity

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The n-connected types are defined inductively:

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X is called $(n+1)$-connected if it is merely inhabited and for all $x, y : X$, the type $x =_X y$ is n-connected.

A map is called n-connected if all its fibers are n-connected types.

2.2. Truncation

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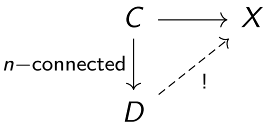
A type X is called n-truncated if it is right-orthogonal to all n-connected maps:



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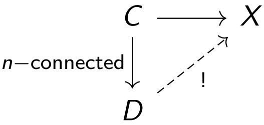

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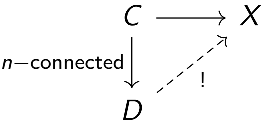

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This is called n-truncation. It satisfies a universal property.

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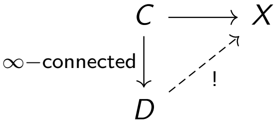

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A map $f : X \to Y$ is n-connected, iff for all $x : X$ and all $k \leq n$:

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Similarly for ∞-connected maps, now for all k.

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So truncatedness gives us a way to express how close we can get to understanding a type just by studying its homotopy groups.

However, we cannot prove that all types are ∞-truncated. This is (famously) independent of HoTT.

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If we can construct such a thing, we will call it ∞-truncation.

Sadly, we do not know how to construct such a thing in HoTT without making further assumptions.

3. Axioms implying ∞-truncation

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Each one of the axioms we'll discuss implies that an ∞-truncation modality can be constructed.

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We already mentioned that this is independent of HoTT. A counter-model is given by the topos of parametrized spectra.

3.2. Postnikov towers

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A Postnikov tower is a sequence of types A_n with maps between them, like so:

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Such that for all k, the kth type in the sequence is k-truncated and the kth map is k-connected.

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Meanwhile, given a Postnikov tower

$$\ldots \longrightarrow A_n \longrightarrow \ldots \longrightarrow A_1 \longrightarrow A_0$$

We can take its limit:

$$\lim A$$

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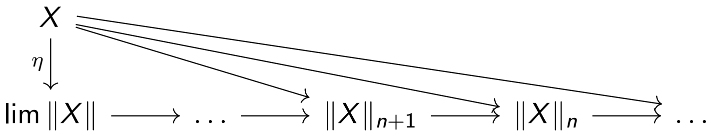
There's a map from a type X to the limit of its Postnikov tower:



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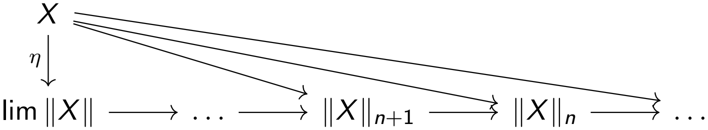

Postnikov convergence says that this vertical map is always an equivalence.

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There are examples of higher toposes which are hypercomplete but not Postnikov convergent.

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Given a Postnikov tower with types $A_0, A_1, \ldots, A_n, \ldots$, we can construct a ladder of maps using the universal property of the n-truncation:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\| \lim A \|_{n+1} & \xrightarrow{\epsilon_{n+1}} & A_{n+1} \\
\downarrow & & \downarrow \\
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Postnikov effectiveness says that all these horizontal maps are always equivalences.

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There are higher toposes which satisfy Postnikov effectiveness, but not hypercompleteness, so this modality is not always trivial.

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To show the converse we need a lemma: if Postnikov effectivness holds, then, for the Postnikov tower of a fixed type, the inverses to the maps ϵ_n in the ladder are the maps $\|\eta\|_n$.

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To show the converse we need a lemma: if Postnikov effectivness holds, then, for the Postnikov tower of a fixed type, the inverses to the maps ϵ_n in the ladder are the maps $\|\eta\|_n$.

Now we appeal to the following diagram:

$$\begin{array}{ccc} X & \longrightarrow & \lim \|X\| \\ \downarrow & & \downarrow \\ \|X\|_n & \xrightarrow{\sim} & \|\lim \|X\|\|_n \end{array}$$

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Anel and Barton [1] generalize this to axioms of countable choice of dimension $\leq d$:

If $X_0, \ldots, X_n, \ldots$ are $(d+k)$-connected objects, then their product is k-connected.

We recover the original case when $d = 0$ and $k = -1$.

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Anel and Barton prove that any one of these forms of countable choice (externally) implies Postnikov effectiveness. They observed that their proof is suitable to be translated into HoTT and formalized.

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`https://github.com/owen-milner/choicepostnikov`

As well as a proof that countable choice implies Postnikov effectiveness, the repository also contains:

- A proof that limits of Postnikov towers are always ∞-truncated
- A proof that Postnikov effectiveness implies that η from above is ∞-connected
- A proof that Postnikov effectiveness implies that the Postnikov operator is a modality

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There are still a few things to add to the formalization.

For example: the equivalence between uniquely eliminating modalities – which are used for the proof – and the modalities already defined in the Cubical library [3].

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The relationship between ∞-truncation and Postnikov towers is subtle in general.

For instance: Morel and Voevodsky [2] present an example of a topos which is hypercomplete, but not Postnikov convergent.

4. Final remarks

Thank you for listening.

[1] M. Anel and R. Barton. "Choice axioms and Postnikov completeness". 2024. URL: https://arxiv.org/abs/2403.19772.

[2] F. Morel and V. Voevodsky. "A^1-homotopy theory of schemes". In: *Publications mathèmatiques de l'I.H.É.S* 90 (1999).

[3] E. Rijke, M. Shulman, and B. Spitters. "Modalities in homotopy type theory". In: *Logical Methods in Computer Science* 16.1 (2020).

Appendix 1. Limits of Postnikov towers are ∞-truncated

Suppose we have a Postnikov tower with types $A_0, A_1, \ldots, A_n, \ldots$, and an ∞-connected map $C \to D$.

The type of fillers for the diagram:

$$\begin{array}{ccc} C & \longrightarrow & \lim A \\ \downarrow & \nearrow & \\ D & & \end{array}$$

is the limit of a diagram whose objects are the types of fillers for diagrams like so:

$$\begin{array}{ccc} C & \longrightarrow & A_n \\ \downarrow & \nearrow & \\ D & & \end{array}$$

And the latter are all contractible because $C \to D$ is n-connected for all n.

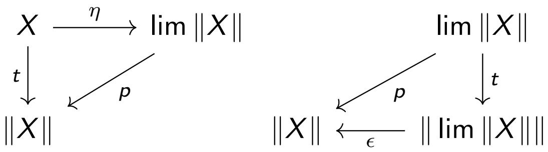
The limit of a diagram of contractible objects is contractible.

Appendix 2. Identifying ϵ_n^{-1}

If Postnikov effectiveness holds, then $\epsilon_n^{-1} = \|\eta\|_n$.

We could write $t_n^X : X \to \|X\|_n$ and $t_n^{\lim \|X\|} : \lim \|X\| \to \| \lim \|X\|\|_n$ for the universal maps. But below we'll supress the sub/superscripts

Then from the definitions of η and ϵ we have some commutative diagrams:



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$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & \lim \|X\| \\
{\scriptstyle t}\downarrow & \swarrow{\scriptstyle p} & \\
\|X\| & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & \lim \|X\| \\
 & {\scriptstyle p}\swarrow & \downarrow{\scriptstyle t} \\
\|X\| & \xleftarrow[\epsilon]{} & \| \lim \|X\| \|
\end{array}
$$

From the second diagram we can deduce that the following diagram commutes

$$
\begin{array}{ccc}
 & & \lim \|X\| \\
 & {\scriptstyle p}\swarrow & \downarrow{\scriptstyle t} \\
\|X\| & \xrightarrow[\epsilon^{-1}]{} & \| \lim \|X\| \|
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$$

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We have:

$$
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 & \swarrow{\scriptstyle p} & \downarrow{\scriptstyle t} \\
\|X\| & \xrightarrow[\epsilon^{-1}]{} & \|\lim \|X\|\|
\end{array}
$$

We can paste these together to arrive at

$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & \lim \|X\| \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\
\|X\| & \xrightarrow[\epsilon^{-1}]{} & \|\lim \|X\|\|
\end{array}
$$

Which shows $\epsilon^{-1} = \|\eta\|$ by the universal property of the truncation.

Appendix 3. Effectiveness implies the Postnikov operator is a modality

We must check that the following map is always an equivalence:

$$\lambda f.f \circ \eta : \left(\prod_{x:\lim \|X\|} \lim \|P(x)\| \right) \to \left(\prod_{x:X} \lim \|P(\eta(x))\| \right)$$

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But, remembering that products commute with limits, and applying the elimination rule for n-truncation, it suffices to check that the following map is always an equivalence for all n:

$$\lambda f.f \circ \epsilon_{n+1}^{-1} : \left(\prod_{x:\| \lim \|X\| \|_{n+1}} Q(x) \right) \to \left(\prod_{x:\|X\|_{n+1}} Q(\epsilon_{n+1}^{-1}(x)) \right)$$

Which is true because ϵ_{n+1}^{-1} is an equivalence.