

Mechanizing Logical Relations

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Mechanization projects

AGDA logrel-mltt by Abel, Öhman, and Vezzosi;

Rocq logrel-coq by Adjedj et al.; McTT by Jang et al.

Quick refresher

Suppose we want to prove canonicity by induction.

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Look at the recursive result on b , return the correct recursive call among n and m .

Harder: higher-order types

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f itself is responsible for the computation.

The recursive call on f should return

$\forall a, P_{\text{Nat}}(n) \rightarrow P_{\text{Nat}}(f n)!$

Canonicity on terms depend on the canonicity on types!

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For $\vdash a : A$, we'd like

$$A_{\text{rel}} : \llbracket A \rrbracket$$

$$a_{\text{rel}} : \llbracket a \rrbracket_{A_{\text{rel}}}$$

And more generally, for $\Gamma \vdash a : A$

$$\Gamma_{\text{rel}} : \llbracket \Gamma \rrbracket$$

$$A_{\text{rel}} : \forall \gamma, (\gamma_{\text{rel}} : \llbracket \gamma \rrbracket_{\Gamma_{\text{rel}}}) \rightarrow \llbracket A[\gamma] \rrbracket$$

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That's the *fundamental lemma* of logical relations.

We also need a corresponding realizer for type and term conversions, $\llbracket A \equiv B \rrbracket$ and $\llbracket a \equiv b \rrbracket_{A_{\text{rel}}}$.

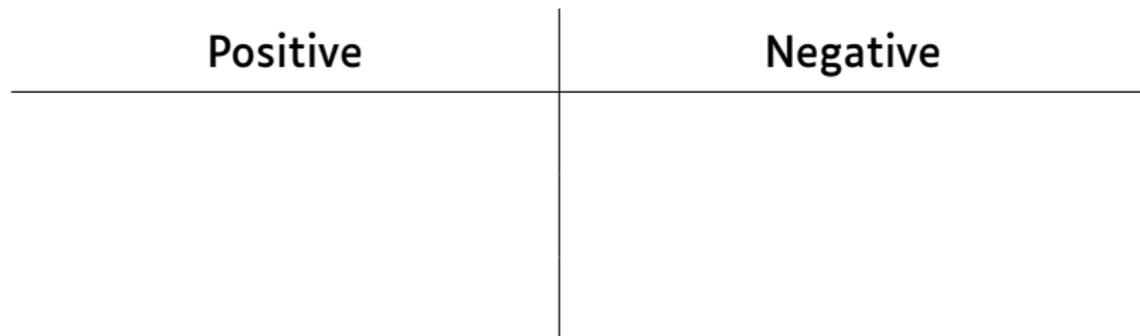
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You can actually save some work and only define conversion realizers as a *partial equivalence relation* with $\llbracket A \rrbracket := \llbracket A \equiv A \rrbracket$.

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We can feed the realizer of $\vdash f : \forall(A : U), A \rightarrow A$ a specific relation to get a parametricity result.

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Positive	Negative
Inductive of codes	Record of relations
Limited to internal types	Can contain external types
Easy to formalize	Dependent PER hell?

Too easy... right?

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But we have no guarantees about correctness! If

$\llbracket b \rrbracket_{\text{Bool}_{\text{rel}}}$ tells me b is true, I want a witness of that!

We could prove correctness after the fact, but it's usually neater to make it correct-by-construction.

→ Let's just add some information in our logical relation!

isTrue :

$\llbracket b \rrbracket_{\text{Bool}_{\text{rel}}}$

$\text{isTrue} : b \rightsquigarrow \text{true} \rightarrow \llbracket b \rrbracket_{\text{Bool}_{\text{rel}}}$

Doesn't give us a derivation.

$\text{isTrue} : \cdot \vdash b \equiv \text{true} \rightarrow \llbracket b \rrbracket_{\text{Bool}_{\text{rel}}}$

Doesn't prove reduction works.

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Axiomatizing precisely the bits we need for the fundamental theorem is difficult.

Instantiating the logical relation

A good candidate for our extra info: derivations for an algorithmic typing system.

$$\frac{\frac{\dots}{\vdash A \rightarrow B \equiv M} \quad \frac{\dots}{\vdash M \equiv A' \rightarrow B'}}{\vdash A \rightarrow B \equiv A' \rightarrow B'} \text{ trans}$$

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The diagram illustrates a derivation path that is being rejected. It starts with a typing derivation (top) and ends with a logical relation derivation (bottom). The path is crossed out with a large red X, and the text "Not so fast!" is written in red next to it.

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Feels unsatisfactory.

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→ Lots of redundant work.

Better served by a form of algorithmic typing followed by reflexivity for algorithmic conversion.

Conclusion

We're far from being able to match the literature on logical relations for theoretical and practical reasons. A lot of refactoring is needed if we want to tackle more advanced systems efficiently.

Thanks for your attention!

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Instead of

$$\frac{\Gamma \vdash A \equiv A' \quad \Gamma, (x : ?) \vdash B}{\Gamma \vdash \Pi x : A. B \equiv \Pi x : A'. B}$$

We have

$$\frac{\Gamma \equiv \Delta \vdash A \equiv A' \quad \Gamma, (x : A) \equiv \Delta, (x : A') \vdash B \equiv B'}{\Gamma \equiv \Delta \vdash \Pi x : A. B \equiv \Pi x : A'. B'}$$

Single mutual inductive

Reify different judgements as an inductive, and index derivations with it.

“ $\Gamma \vdash t : A$ ” : judgement
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Avoids Combined **Scheme** and meta-programming!