Realizability Triposes from Sheaves

Bruno da Rocha Paiva Vincent Rahli

University of Birmingham

June 2025

Motivation

Choice sequences were originally introduced in Brouwer's second act of intuitionism [2]:

- they are infinite sequences whose values are generated in a possibly nondeterministic manner;
- we only ever have access to a finite number of values.

They are anti-classical but with them Brouwer gave a successful account of analysis in an intuitionistic setting.

In previous work we mixed them with a realizability model of type theory to separate three different versions of Markov's principles [1].

Choice sequence axioms

Assuming we have a type *ChoiceSeq* of choice sequences.
Each element $\delta : \mathit{ChoiceSeq}$ be coerced to a function $\delta : \mathbb{N} \to \mathbb{N}$.

- **Density Axiom**:
  For every list l of natural numbers, there exists a choice sequence δ with l as a prefix.

- **Decidability of Equality**:
  Equality of elements in *ChoiceSeq* is decidable.

- **Axiom of Open Data**:
  Given a predicate $P : \mathit{ChoiceSeq} \to \Omega$, if $P\delta$ holds then there exists some natural number n such that for all $\sigma : \mathit{ChoiceSeq}$ which agree with δ on their first n entries, $P\sigma$ also holds.

Setting the table

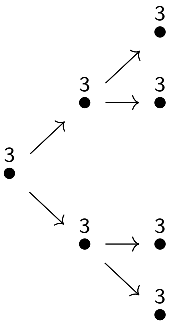
Choice sequences are usually treated formally through Kripke/Beth style semantics $\rightsquigarrow$ leads to presheaves and sheaves.

Fix a rooted tree $\mathbb{W}$ seen as a poset of worlds.

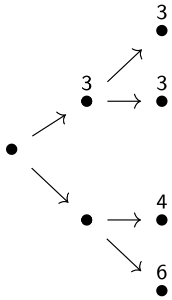
Given a world $w : \mathbb{W}$ and an upwards closed subset $U \subseteq \mathbb{W}$, we say that U **covers** w if all paths through $\mathbb{W}$ which start from w, eventually reach U.

With sheafification come choice sequences

Pure natural numbers:



If you sheafify then you allow:



The function space $\Delta\mathbb{N} \to a\Delta\mathbb{N}$ features similar notion of nondeterminism as choice sequences (but misses the previous axioms).

A first attempt at a sensible tripos

Start with a pca A with application $- \cdot_w -$ indexed by $w : \mathbb{W}$.

Given a presheaf X, we define **realizability predicates on** X as natural transformations from X to $\mathcal{P}_{\square}(\mathsf{A})$

$a \in \varphi_w(x)$ means that a is evidence that x satisfies φ at world w

We want to order predicates: say that $\phi \leq \psi$ at world w if there exists a code $e : \mathsf{A}$ such that for all extensions $u \leq w$, elements $x : X_u$ and codes $a : \mathsf{A}$, if $u \in \phi_u(x, a)$ then there exists a cover $\mathcal{V}$ of u such that for all $v \in \mathcal{V}$ we have

$$e \cdot_v a \downarrow \quad \text{and} \quad v \in \psi_v(x|_v, e \cdot_v a)$$

Avoiding explicit mention of covers

We can use a Lawvere-Tierney topology to avoid explicit mention of covers.

That is a modality $\square : \Omega \to \Omega$ such that

- $P \Rightarrow \square P$
- $\square\square P \Rightarrow \square P$
- $\square(P \wedge Q) = \square P \wedge \square Q$

A definition internal to a topos

Assume we have an internal pca A in $\mathcal{E}$.

Given an object X of $\mathcal{E}$, we define **realizability predicates on** X as the type $X \to \mathcal{P}_\Box(\mathsf{A})$.

We can order realizability predicates, we say $\varphi \leq \psi$ if we have a uniform way of sending evidence of φ to evidence for ψ:

$$\exists e : \mathsf{A}. \forall x : X. \forall a \in \varphi(x). \Box(e \cdot a \downarrow \wedge e \cdot a \in \psi(x))$$

This extends to give a tripos T on $\mathcal{E}$ [3].

Next steps

- Can define a geometric morphism from $\mathcal{E}_{\square}[T] \to \mathcal{E}[T]$ which gives an analogue of sheafification on $\mathcal{E}[T]$.
  $\rightsquigarrow$ sends a type to an effectful version where elements may depend on the world in a realizable way.
- Different presheaf pcas should be able to validate the different choice axioms.
- Do we lose anything from the computational type theory setting? Can we still separate different versions of Markov's principle?

References

Yannick Forster, Dominik Kirst, Bruno da Rocha Paiva, and Vincent Rahli.
Markov's Principles in Constructive Type Theory.
In *29th International Conference on Types for Proofs and Programs*, Valencia, Spain.

Mark van Atten.
On Brouwer.
Wadsworth Publishing Company, 2004.

Jaap van Oosten.
Exercises in Realizability.
Doctoral, University of Amsterdam, August 2018.