Rezk Completions For (Elementary) Topoi

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Abstract

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In this work, we *lift* this completion to topoi.

1. Introduction

2. Goal + Approach

Univalent Categories: Definition

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A category $\mathcal{C}$ is **univalent** if

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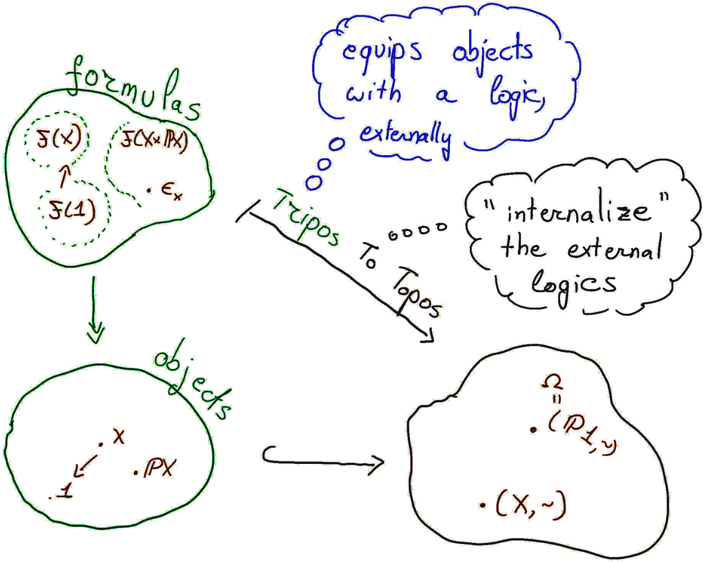
3. Univalent categories are particularly well-behaved:
   - Notions unique up to isomorphism become unique up to identity;
   - Isomorphisms between univalent categories coincide with equivalences.

Tripos-To-Topos



Univalent Categories: Non-Examples

The following category is the topos coming from the trivial Set-tripos.

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The category of sets and trivial hom-sets is not univalent.

Hence, even if we assume univalence of the underlying base category (of the tripos), the resulted topos is in general not univalent.

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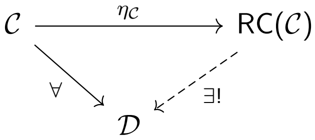
There are 2 sufficiently good implementations of $\mathrm{RC}(\mathcal{C})$.

The Universality of The Rezk Completion: part 1

Let $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ be the Rezk completion.



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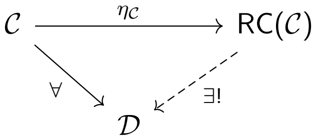

Universal Property: 1-categorically

For every univalent category $\mathcal{D}$,

$$(\eta_{\mathcal{C}} \cdot -) : [\mathrm{RC}(\mathcal{C}), \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]$$

is an equivalence of categories.

The Universality of The Rezk Completion: part 2

Assuming $\mathrm{RC}(\mathcal{C})$ is given for all categories $\mathcal{C}$:

Universal Property: 2-categorically

The Rezk completion inclusion $\mathsf{Cat}_{\mathsf{univ}} \xhookrightarrow{\iota} \mathsf{Cat}$ admits a left biadjoint RC.

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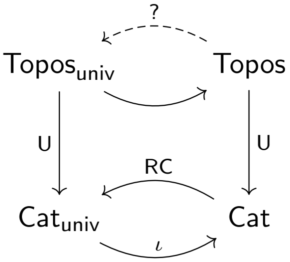
Lift the biadjunction RC $\dashv \iota$ to bicategories of topoi, or more generally: bicategories of *structured categories*.



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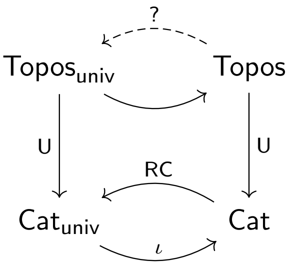

To solve the goal, we take 2 steps.

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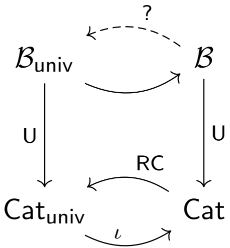
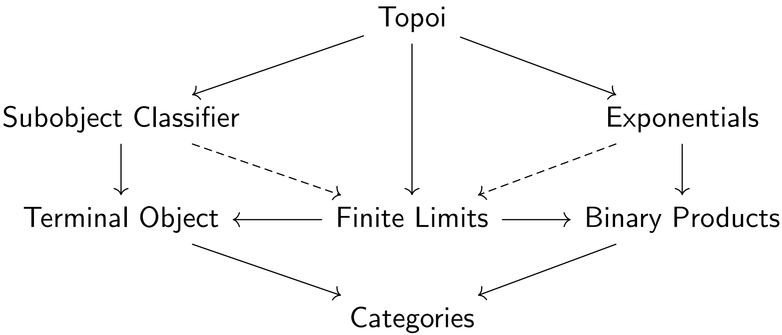
Let $\mathsf{U} : \mathcal{B} \to \mathsf{Cat}$ be a forgetful pseudofunctor, and $\mathcal{B}_{\mathsf{univ}} \to \mathsf{Cat}_{\mathsf{univ}}$ the pullback along $\mathsf{Cat}_{\mathsf{univ}} \hookrightarrow \mathsf{Cat}$.



Tower Of Topos Structure



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Main lemma

$\mathcal{B}_{\mathsf{univ}} \hookrightarrow \mathcal{B}$ has a left biadjoint if for every weak equivalence $G : \mathcal{C}_0 \to \mathcal{C}_1$ with $\mathcal{C}_1$ univalent:

1. for every $x : \mathrm{U}^{-1}(\mathcal{C}_0)$, there are $\hat{x} : \mathrm{U}^{-1}(\mathcal{C}_1)$

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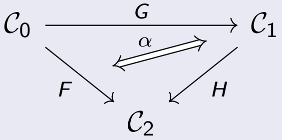

with $\mathcal{C}_2 : \mathrm{Cat}_{\mathrm{univ}}$, $x_i : \mathrm{U}^{-1}(\mathcal{C}_i)$:

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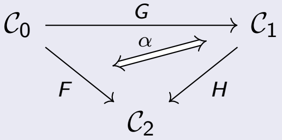

with $\mathcal{C}_2 : \mathrm{Cat}_{\text{univ}}$, $x_i : \mathrm{U}^{-1}(\mathcal{C}_i)$:

for every $f : x_0 \to_F x_2$, there is given $\hat{f} : x_1 \to_H x_2$.

RC for elementary topoi

Analogously as above:

Lemmata

The following structures on a category are compatible with Rezk completions:

1. Finite (co)limits;
2. Subobject classifiers;
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Theorem

The inclusion $\mathsf{Topos}_{\mathsf{univ}} \hookrightarrow \mathsf{Topos}$ has a left biadjoint.

Future Directions

1. RC for other structures: LCCC, extensive;
2. Interaction internal logic with RC;
3. Computing *concrete* Rezk completions: Higg's theorem; Assemblies;
4. ...

Conclusion

We have formalized, in UniMath:

1. Displayed universal arrows;
2. The lifting of RC to the aforementioned structures;
3. The tripos-to-topos construction.

Take-aways:

1. Taking Rezk completions is necessary for some constructions;
2. Rezk completions commute with a lot of structure, but not all;
3. Displayed and bicategorical methods provide a suitable level of abstraction.

Terminal Slide

Any questions, remarks, $\cdots$?