Verifying Z3 RUP Proofs with the Interactive Theorem Provers Coq/Rocq and Agda

Harry Bryant, Andrew Lawrence,
Monika Seisenberger & Anton Setzer
Swansea University

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Contributors

Harry Bryant (PhD student)
(Heavy weight lifting + Slides design)

Andrew Lawrence (Siemens Mobility)
(Money + Industrial expertise)

Monika Seisenberger (PhD supervisor)
(Money + Lead)

Anton Setzer (PhD co-supervisor)
(Dependent type missionary + Presenter)

Motivation

- **Joint work with Siemens Mobility:** Verification of railway interlocking systems
- **High safety requirements:** Safety-critical infrastructure
- **Why Z3?** Widely used SMT solver,
  Required by Siemens because industrial tool (Microsoft Research; liability issues).
- **Challenge:** SMT solvers can produce incorrect results
- **Community response:** SAT conferences now require proof checkers.
- **Safety Critical Systems** requires much higher level of correctness of proof checker than mathematics.
- Main problem **correctness of actual implementation of proof checker** rather than theoretical algorithm.

Key Insight

For safety-critical systems, we need **verified checkers**.

Verification doesn't replace Testing . . .

- Would you fly plane which has been **fully verified in Agda**

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- but **never been flown?**

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- Example of small toy railway interlocking system developed by Anton
- fully verified but trains started to disappear.
- Disappearing trains happened in real world interlocking systems (US)

. . . but can reduce cost

- **Reduces cost of testing** (finding errors earlier)
- Find some errors thorough testing won't find.

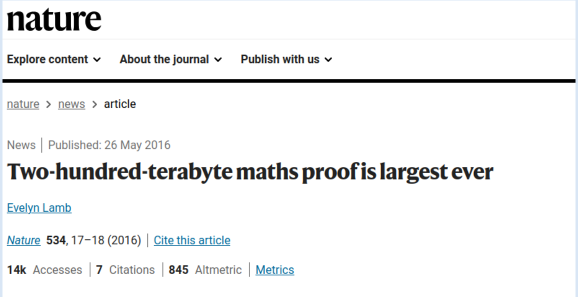
Industrial Use of Formal Methods

- **Big progress in use of formal methods**
  Now high level discussions about limits of SMT solving and Rocq prover possible.
- **Tool chain** in railway verification [BCL+23].
- Need industrial tools **licensed or under control of Siemens**.

SMT Proofs can be very long

Heule-Kullmann-Markek's "largest proof in the world"

200 TB maths proof [Lam16, HKM16].
Used already DRAT format (based on RUP) [FHB+24]
Generated on a supercomputer in Texas
In this form never made it to Swansea.
Compressed proof: 68 GB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Evelyn Lamb

Nature **534**, 17–18 (2016) | Cite this article

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SMT Proofs can be very long

Why It Matters?

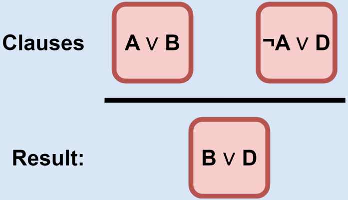
Proof size is a real challenge:

- Old resolution proofs not feasible.
- Verifying them requires significant computational resources.
- Proof checking more complex – proof checker requires verification.
- **Industrial proofs:** smaller size but still big and much higher requirement on correctness assurance.
- Need for efficient, trustworthy tools for handling large-scale proofs

Proofs in Z3

Old Z3 Proof Format

Based on full resolution.
Easy to verify.
Size problem.



New Z3 Proof Format based on RUP

- One of several more compressed proof proof formats for SAT/SMT solving.
- Introduced September 2022 for Z3 proofs [Bjø22].
- Based on Conflict Driven Clause Learning.
- More difficult to check and verify correctness of proof checker.

Our Solution

Project Plan

- **Prototype proof checker in Agda** including correctness proof.
- Write proof checker in Rocq.
- **Prove correctness in Rocq**.
- **Correctness doesn't require** creating **tree proofs** (resource consuming).
  - Optional creation of tree proofs for additional confidence
- **Extract verifier** from Rocq as **efficient C-program**.
- **Verifier is extensible** to addition of additional SMT features.
- **Two-/Three-level proof pattern** for proof of correctness of proof calculi in dependent type theory
- Use it for **integrating Z3 proofs into Agda** (Work in progress).

1. Introduction
2. Background
3. Correctness
4. Two vs Three Level Approach
5. Conclusion

1 Introduction

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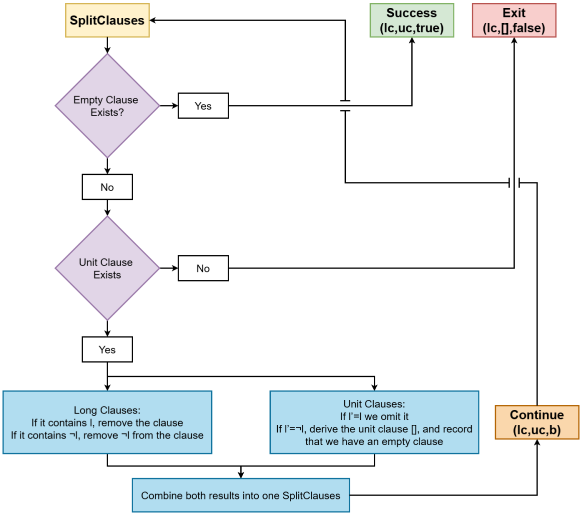
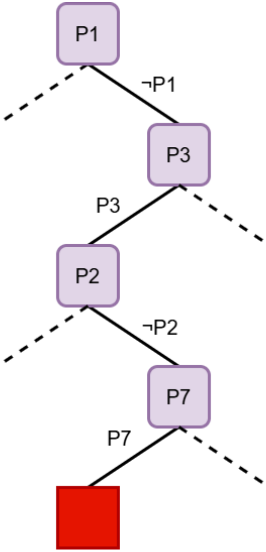
SAT Solving - General Setup

- We work on SAT-solving part of SMT solving.
- Basis propositional variables (which may denote longer SMT formulas).
- **Literals:** $l_1, l_2, \ldots$ Positive or negative propositional variables
- **Clauses:** Disjunctions of literals written $c = [l_1, l_2, l_3]$
- **Sequents:** Conjunctions of clauses.
- Split sequents using **Clause Splitting:**
  - Long clauses ($length \geq 2$)
  - Unit clauses ($length = 1$)
  - Empty clause (contradiction)
- Represented as: `SplitClauses = LongClauses × UnitClauses × Bool`
- **Bool flag:** `true` means empty clause exists (successful proof).

Unit-Clause Propagation – Inexpensive Reductions



Conflict Driven Clause Learning



- $\neg P1 \wedge P3 \wedge \neg P2 \wedge P7 \rightarrow$ conflict.
- Deeper analysis optimises it, e.g. $(P3 \wedge P7) \rightarrow$ conflict
- Therefore, add conflict clause $[\neg P3, \neg P7]$
- Backtrack to decision level $P3$ and choose $\neg P3$

Reverse Unit Propagation (RUP)

RUP Inference

A clause $C = [l_1, l_2, \ldots, l_k]$ is a RUP Inference from a formula F if:

The unit clauses $[\neg l_1], [\neg l_2], \ldots, [\neg l_k]$, when added to F, make the formula refutable via Unit-Clause Propagation (UCP).

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RUP Proof

A sequence of clauses $C_1, C_2, \ldots,$ where each C_i is a RUP Inference from the formula:

$$F_j = F_{j-1} \cup \{C_j\}, \quad j \geq 1.$$

If a clause is a RUP Inference, its negation will lead to a contradiction via UCP.

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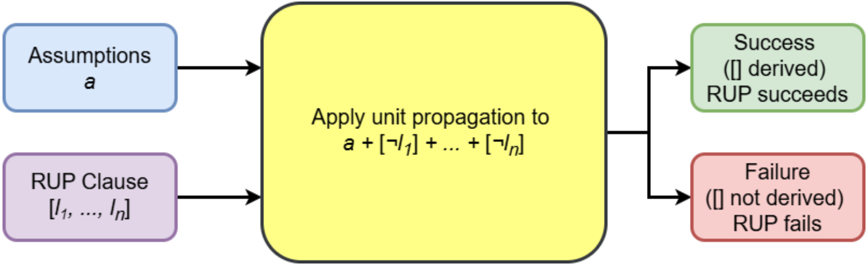
RUP Refutation

A RUP Proof in which some clause $C_j = []$. This indicates that F_0 is unsatisfiable.

RUP Checker

For each RUP Inference, apply the RUP Checker to the list of assumptions *a*:



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Proof of Correctness

Lemma: $A \vdash_{\mathrm{UnitResolution}} \mathrm{UnitProp}(A)$.

Proof: Use unit resolution to derive from
$[l_1, \ldots, l_{n-1}, \neg l]$ and $[l]$
$[l_1, \ldots, l_{n-1}]$.

Lemma: If $A \vdash_{\mathrm{UnitResolution}} B$ then $A \models B$.

Proof of Correctness

Lemma: If $\texttt{RUPChecker}(A, [l_1, ..., l_n])\ \texttt{=}\ \texttt{true}$, then:

$$A + [\neg l_1] + \cdots + [\neg l_n] \models []$$

- $\texttt{RUPChecker}(A, [l_1, ..., l_n])\ \texttt{=}\ \texttt{true}$
- $\Rightarrow A + [\neg l_1] + \cdots + [\neg l_n] \vdash_{\text{UnitResolution}} []$
- $\Rightarrow A + [\neg l_1] + \cdots + [\neg l_n] \models []$.

Proof of Correctness

Lemma: One step entailment from conflict

$$A + [\neg l] \models c \quad \Rightarrow \quad A \models [l] \cup c$$

Lemma: Entailment from Conflict

$$A + [\neg l_1] + \cdots + [\neg l_n] \models [] \quad \Rightarrow \quad A \models [l_1, \ldots, l_n]$$

Theorem: Soundness of RUP Checker

$$\mathtt{RUPChecker}(A, c) = \mathtt{true} \Rightarrow A \models c$$

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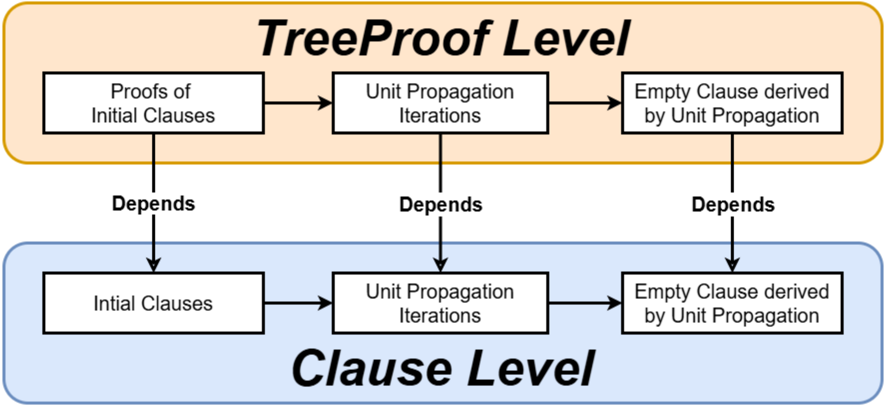
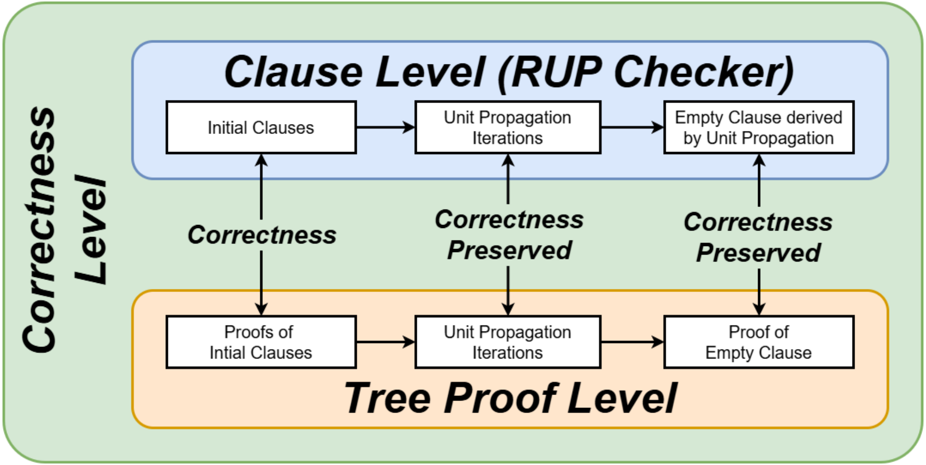
Agda: Two Level Approach for Correctness



Dealing with Resistance to Dependent Types

- Facing rebellion against dependent types
  (by Swansea's logic group)
- Therefore smuggling in dependent types where acceptable:
  - Treeproofs depending on clauses not acceptable.
  - Dependent correctness predicate is acceptable.

Rocq: Three Level Approach for Correctness



Example Three Level Approach (Rocq)

```
Definition SplitClauses : Type := (list Clause * list Literal * bool).

Definition SplitTreeProofs : Type := (list TreeProof * list TreeProof * option TreeProof).

Definition CorrectSplit (al : Assumption)(c : SplitClauses)
          (t : SplitTreeProofs) : Prop :=
  match c with
  | (clauses,literals,b) =>
      match t with
      | (ct,lt,bt) => (CorrectProofList al clauses ct) /\
                      (CorrectLiteralProof al literals lt) /\
                      (CorrectOptionTreeProof' al b bt)
      end
  end.
```

Example ProofStep (Three Level Approach)

```
Definition propagationStep
  (clauses : list Clause)
  (literals : list Literal)
  (l : Literal) : SplitClauses :=
  combineSplitClausesSplitLits (processAndSplitClausesWithLit clauses l)
    (processListLitsWithLit literals l).

Definition propagationStepProofs (clauses : list Clause)
  (literals : list Literal) (l : Literal) (proofs_c proofs_l : list TreeProof)
  (tp : TreeProof) : SplitTreeProofs :=
  combineSplitTreeProofs (process_and_extract_treeproofs clauses l proofs_c tp)
    (remove_treeproof literals proofs_l l tp).
```

Example ProofStep (Three Level Approach)

```
Lemma propagationStepCorrect :
  forall (al : Assumption)
         (clauses : list Clause)
         (literals : list Literal)
         (l : Literal)
         (proofs_c proofs_l : list TreeProof)
         (tp : TreeProof),
    CorrectProofList al clauses proofs_c ->
    CorrectLiteralProof al literals proofs_l ->
    CorrectProof al [l] tp ->
    CorrectSplit al
      (propagationStep clauses literals l)
      (propagationStepProofs clauses literals l proofs_c proofs_l tp).
```

Theorems in Agda and Rocq

Theorem in Agda:

```
rupCorrect : (f : Formula)(rp : Clause) → (atom (checkOneRup f rp))
                → EntailsCl f rp
```

Theorem in Rocq:

```
(* Main Theorem *)
Lemma RUP_Checker_correct :
  forall (a : Assumption)(c : Clause),
    RUP_Checker a c = true -> entails a c.
```

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Conclusion

- Addressing RUP format of proofs.
- **Theorem RUPChecker**$(A, c) =$ **true** $\Rightarrow$ **A** $\models$ **c.**
- Proofs in Agda and Rocq.
- **Two-** and **three level approach to proving correctness**
  More general proof pattern.
- No need to generate tree proofs from RUP proof (resource consuming) and then check them.
  - But option to compute tree proofs of [] if RUPChecker(a,c) returns true for extra confidence.
- Proof checker works well on railway examples.
  - smaller proofs: 150,000 lines, roughly 30,000 steps, 3 mins.
  - larger proofs: 4,750,000 lines, roughly 500,000 steps, 7.5 hrs.
- Should allow to integrate output from SAT solvers into Agda and Rocq proofs (Important for Agda!).
  - Combination of interactive and interactive theorem proving.
  - Modelling interactively, verification conditions using SMT solving.

Limitations

- Need verified extraction of programs from Rocq.
- Need to explore use of trusted core of Rocq.

Thank You for Listening

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