Yet another homotopy group, yet another Brunerie number

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- Why do we care?

Why do we care about $\pi_5(\mathbb{S}^3)$?

We don't... We actually care about $\pi_6(\mathbb{S}^4)$, the second stable homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}
$\mathbb{S}^1$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
$\mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{S}^3$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{S}^4$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$
$\mathbb{S}^5$	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{S}^6$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$
$\mathbb{S}^7$	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0
$\mathbb{S}^8$	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$

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- It turns out that $\pi_5(\mathbb{S}^3) \cong \pi_6(\mathbb{S}^4)^*$ and the former is easier to compute directly

*Follows from the quaternionic Hopf fibration (Buchholtz & Rijke, 18)

Our work is a natural continuation of Brunerie's proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. His strategy:

1. Show that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z}$ for some $n : \mathbb{Z}$
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- More on this soon – first, let's see what this n comes from

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Basic definitions

- The only higher inductive types we'll need in this talk are cofibres:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & C_f \end{array}$$

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- With this, we can define the nth homotopy group of a pointed type A. We set

$$\pi_n(A) := \| \mathbb{S}^n \to_\star A \|_0$$

The pinch map

- For any map $f : A \to B$, there is a function $\mathrm{pinch}_f : C_f \to \Sigma A$ defined as follows

$$C_f = \text{Pushout of:} \qquad 1 \longleftarrow A \xrightarrow{f} B$$

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$$

- The technical content of our proof is really concerned with the long exact sequence of pinch_f:

$$\cdots \to \pi_{n+1}(\Sigma B) \to \pi_n(\mathsf{fib}_{\mathsf{pinch}_f}) \to \pi_n(C_f) \to \pi_n(\Sigma B) \to \pi_{n-1}(\mathsf{fib}_{\mathsf{pinch}_f}) \to \dots$$

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- Question answered by our technical theorem: when can we swap $\pi_n(\mathrm{fib}_{\mathrm{pinch}_f})$ for something nicer?
  - Answer: when f is a Whitehead product

Whitehead products

Fact

Given pointed functions $f : \mathbb{S}^n \to_\star A$ and $g : \mathbb{S}^m \to_\star A$, there is a function $[f, g] : \mathbb{S}^{n+m+1} \to_\star A$ called the *Whitehead product* of f and g.

- Can be viewed as a bilinear multiplication $[-,-] : \pi_n(A) \times \pi_m(A) \to \pi_{n+m+1}(A)$

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- Can be viewed as a bilinear multiplication $[-,-] : \pi_n(A) \times \pi_m(A) \to \pi_{n+m+1}(A)$
- The original Brunerie number was defined in terms of Whitehead products

Brunerie's theorem (2016)

Let η denote the canonical generator of $\underbrace{\pi_3(\mathbb{S}^2)}_{\cong \mathbb{Z}}$. We have that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z}$ for the $n : \mathbb{Z}$ satisfying $[\mathrm{id}_{\mathbb{S}^2}, \mathrm{id}_{\mathbb{S}^2}] = n \cdot \eta$.

- We will prove an almost identical result for $\pi_5(\mathbb{S}^3)$

The main technical theorem

The key technical result:

Main technical theorem (demo version)

Let $f : \pi_n(\mathbb{S}^m)$. We have $\pi_{2n}(C_{[\mathrm{id}_{\mathbb{S}^m}, f]}) \cong \pi_{2n}(\mathrm{fib}_{\mathrm{pinch}_f})$.

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- (For those who care, here's the full result)

Main technical theorem (full version for generalised Whitehead products)

Let A be an $(a-1)$-connected pointed type, B be any pointed type and let $f : \Sigma A \to_\star \Sigma B$. In this case, there is a $2a$-connected map $\gamma : C_{[\mathrm{id}_{\Sigma B}, f]} \to \mathrm{fib}_{\mathrm{pinch}_f}$.

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- We will apply the lemma in the case when $f = [\mathrm{id}_{\mathbb{S}^2}, \mathrm{id}_{\mathbb{S}^2}] : \mathbb{S}^3 \to \mathbb{S}^2$

Applying the main theorem

$$\pi_5(\mathbb{S}^4) \longrightarrow \pi_4(\mathsf{fib}_{\mathsf{pinch}_f}) \longrightarrow \pi_4(C_f) \longrightarrow \pi_4(\mathbb{S}^4) \longrightarrow \pi_3(\mathsf{fib}_{\mathsf{pinch}_f}) \longrightarrow \pi_3(C_f)$$

Applying the main theorem

$$\pi_5(\mathbb{S}^4) \longrightarrow \pi_4(\mathsf{fib}_{\mathsf{pinch}_f}) \longrightarrow \pi_4(C_f) \longrightarrow \pi_4(\mathbb{S}^4) \longrightarrow \pi_3(\mathsf{fib}_{\mathsf{pinch}_f}) \longrightarrow \pi_3(C_f)$$

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You stare at this sequence...

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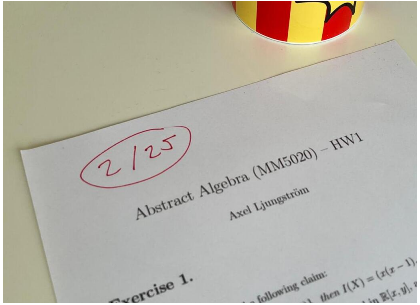

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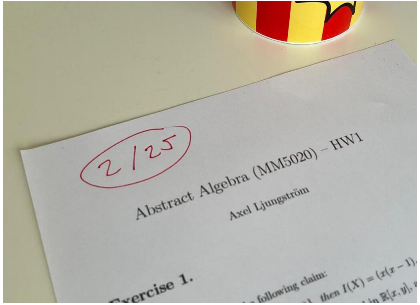

- The number $d(1)$ is obtained by applying the isomorphism $\pi_4(\mathbb{S}^2) \cong \mathbb{Z}/2\mathbb{Z}$ to the composite map $\mathbb{S}^4 \xrightarrow{\Sigma\eta} \mathbb{S}^3 \xrightarrow{[\mathrm{id}_{\mathbb{S}^2}, \mathrm{id}_{\mathbb{S}^2}]} \mathbb{S}^2$ (viewed as an element of $\pi_4(\mathbb{S}^2)$)
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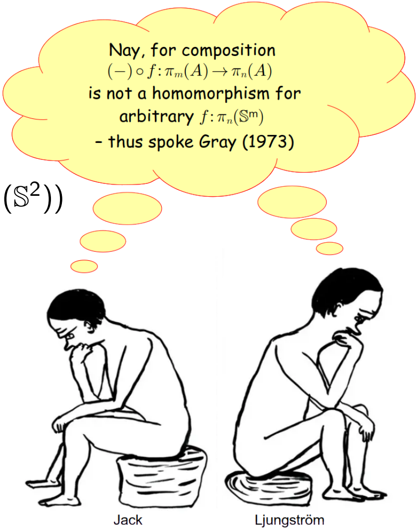

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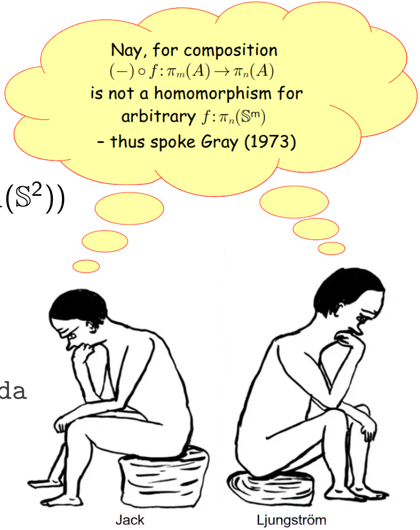

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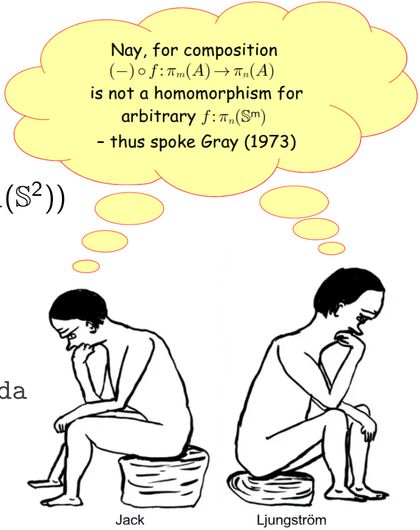
  - It doesn't



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Normalising $d(1)$

- What is actually happening here? The isomorphism $\pi_4(\mathbb{S}^2) \cong \mathbb{Z}/2\mathbb{Z}$ appearing in the definition of $d(1)$ consists of two problem makers:

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Bad guy 2

- This isomorphism is implicitly constructed in terms of *the proof that* the original Brunerie number has absolute value 2.
- *At least* as difficult as computing the original Brunerie number

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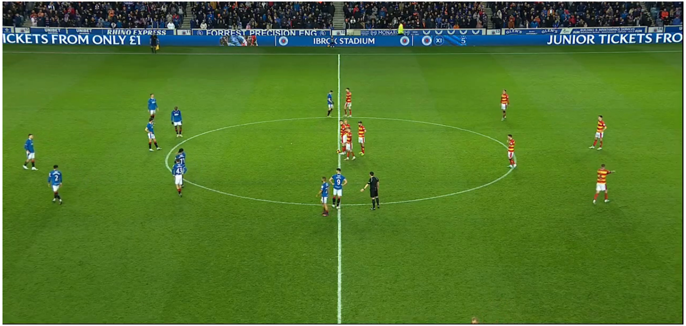
A last minute go at the pen-and-paper proof

- Close to the revision deadline for TYPES2025, we decided to give the pen-and-paper proof another shot
- Suddenly, the resistance had changed...



A last minute go at the pen-and-paper proof

- Recall: we would be done if we could show that $(-) \circ f : \pi_m(A) \to \pi_n(A)$ is a homomorphism for $f : \pi_n(\mathbb{S}^m)$
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- So, in a somewhat anti-climactic way, we have shown that $\pi_5(\mathbb{S}^3) \cong \mathbb{Z}/2$
  $\implies \pi_{n+2}(\mathbb{S}^n) \cong \mathbb{Z}/2$ for $n \geq 3$

Conclusions

- Just like Brunerie, we proved that
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- Stay tuned for $\pi_8(\mathbb{S}^5) = \mathbb{Z}/n\mathbb{Z}$ (maybe this n will be more exciting)

Thanks for listening!

References

[1] Guillaume Brunerie. "On the homotopy groups of spheres in homotopy type theory". PhD thesis. Université Nice Sophia Antipolis, 2016. arXiv: 1606.05916.

[2] Brayton Gray. "On the Homotopy Groups of Mapping Cones". In: *Proceedings of the London Mathematical Society* s3-26.3 (Apr. 1973), pp. 497–520.

[3] Brayton Gray. "Filtering the fiber of the pinch map". In: *Groups, homotopy and configuration spaces (Tokyo 2005)*. Mathematical Sciences Publishers, Feb. 2008, pp. 203–227.

[4] Ulrik Buchholtz and Egbert Rijke. "The Cayley-Dickson Construction in Homotopy Type Theory". In: *Higher Structures* 2.1 (2018), pp. 30–41.

[5] Axel Ljungström. *Some properties of Whitehead products*. Extended abstract at *Workshop on Homotopy Type Theory/Univalent Foundations (HoTT/UF 2025)*. 2025. URL: https://hott-uf.github.io/2025/abstracts/HoTTUF_2025_paper_23.pdf.

[6] Sephora. *Best Products For Whiteheads*. 2025. URL: https://www.sephora.com/buy/best-products-for-whiteheads.

[7] Axel Ljungström and Anders Mörtberg. *Formalising and Computing the Fourth Homotopy Group of the 3-Sphere in Cubical Agda*. 2024. arXiv: 2302.00151.