About the construction of simplicial and cubical sets in indexed form: the case of degeneracies

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The problem of *semi-simplicial types*

Construct the following family of family types in homotopy type theory:

$$X_0 \;:\; \mathsf{Type} \qquad \text{(points)}$$

$$X_1 \;:\; X_0 \times X_0 \to \mathsf{Type} \qquad \text{(segments)}$$

$$X_2 \;:\; \Pi x_1 x_2 x_3.\, X_1(x_1, x_2) \to X_1(x_1, x_3) \to X_1(x_2, x_3) \to \mathsf{Type} \qquad \text{(triangles)}$$

$$\vdots$$

The problem of *semi-simplicial types*

Construct the following family of family types in pure type theory:

$X_0 \;:\; \mathsf{Type}$ (points)

$X_1 \;:\; X_0 \times X_0 \to \mathsf{Type}$ (segments)

$X_2 \;:\; \Pi x_1 x_2 x_3.\, X_1(x_1, x_2) \to X_1(x_1, x_3) \to X_1(x_2, x_3) \to \mathsf{Type}$ (triangles)

$\vdots$

two problems into one

↙ ↘

semi-simplicial *types*
(needs – at least – a description of higher-dimensional coherences)

recipe for *indexed* semi-simplicial
(this talk, for **HSet**, see also Voevodsky, Part-Luo, Altenkirch-Capriotti-Kraus)

The fibred/indexed correspondence for **HSet**

For B : HSet

$$\begin{array}{c} E : \mathsf{HSet} \\ \Big\downarrow \\ B \end{array} \quad \simeq \quad B \to \mathsf{HSet}$$

Iterating the fibred/indexed correspondence for **HSet**

Application to definition of Reedy presheaves in *indexed form*, here for semi-cubical sets:

fibred form $\quad vs \quad$ *indexed form*

$$
\begin{array}{ll}
Y_0 : \mathsf{HSet} & X_0 : \mathsf{HSet} \\
\uparrow d^L \; \uparrow d^R & \\
Y_1 : \mathsf{HSet} & X_1 : X_0 \times X_0 \to \mathsf{HSet} \\
\uparrow d^{L\star} \; \uparrow d^{R\star} \; \uparrow d^{\star L} \; \uparrow d^{\star R} & \\
Y_2 : \mathsf{HSet} & X_2 : \Pi(x_{LL}, x_{LR}).\, \Pi x_{L*} : X_1(x_{LL}, x_{LR}). \\
 & \qquad \Pi(x_{RL}, x_{RR}).\, \Pi x_{R*} : X_1(x_{RL}, x_{RR}). \\
+ \text{ coherences} & \qquad X_1(x_{LL}, x_{RL}) \times X_1(x_{LR}, x_{RR}) \to \mathsf{HSet}
\end{array}
$$

Iterating the fibred/indexed correspondence for **HSet**

Application to the definition of Reedy presheaves in *indexed form*, here for cubical sets:

fibred form		*vs*	*indexed form*	
Y_0	$:$ **HSet**		X_0	$:$ **HSet**
$\uparrow d^L \; \uparrow d^R$				
Y_1	$:$ **HSet**		X_1	$: X_0 \times X_0 \to$ **HSet**
$\uparrow d^{L\star} \; \uparrow d^{R\star} \; \uparrow d^{\star L} \; \uparrow d^{\star R}$				
Y_2	$:$ **HSet**		X_2	$: \Pi(x_{LL}, x_{LR}).\, \Pi x_{L*} : X_1(x_{LL}, x_{LR}).$ $\Pi(x_{RL}, x_{RR}).\, \Pi x_{R*} : X_1(x_{RL}, x_{RR}).$ $X_1(x_{LL}, x_{RL}) \times X_1(x_{LR}, x_{RR}) \to$ **HSet**
$+$ coherences				

Motivations:

1. The iterated fibred/indexed correspondence is interesting in itself

2. Suggests models of type theory closer to the syntax: e.g. equality interpreted as a (relevant) relation rather than as a span

Rest of the talk

Presheaves in “indexed” form

- following a n-ary “parametricity” recipe

    - s.t. *unary* parametricity gives augmented semi-simplicial sets

    - and *binary* parametricity gives semi-cubical sets

- equipped with a *degeneracy*

- machine-checked in Rocq

A uniform approach to augmented simplicial sets and cubical sets

Augmented simplicial and cubical categories only differ in the "arity" of a finite set ν:

$$
\begin{array}{ll}
\mathsf{Obj} & := \mathbb{N} \\
\mathsf{Hom}(p,n) & := \{l \in (\nu \sqcup \{\star\})^n \mid \text{number of } \star \text{ in } l = p\} \\
g \circ f & := \begin{cases} f & \text{if } g = \epsilon \\ a\,(g' \circ f) & \text{if } g = a\,g', \text{where } a \in \nu \\ a\,(g' \circ f') & \text{if } g = \star\,g',\ f = a\,f', \text{ where } a \in \nu \text{ or } a = \star \end{cases} \\
\mathsf{id} & := \star \ldots \star\ \ n \text{ times for } \mathsf{id} \in \mathsf{Hom}(n,n)
\end{array}
$$

A uniform approach to augmented simplicial sets and cubical sets

Augmented simplicial and cubical categories only differ in the "arity" of a finite set ν:

$$\begin{aligned}
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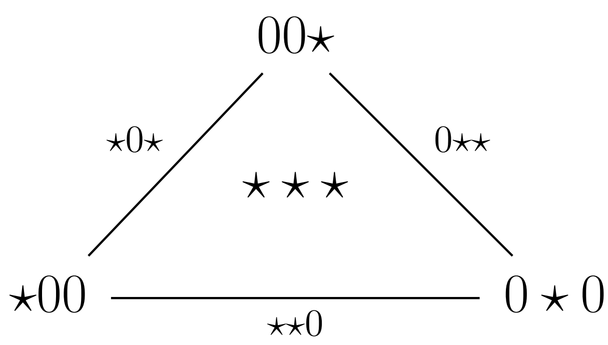
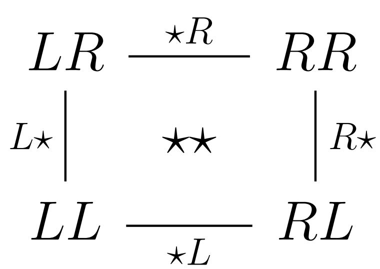
\mathsf{id} &:= \star \ldots \star\ \ n \text{ times for } \mathsf{id} \in \mathsf{Hom}(n,n)
\end{aligned}$$

augmented semi-simplicial sets with $\nu = \{0\}$ (counting from -1)

semi-cubical sets with $\nu = \{L, R\}$ (counting from 0)





An effective indexed construction as a dependent stream of dependent sets

ν-sets

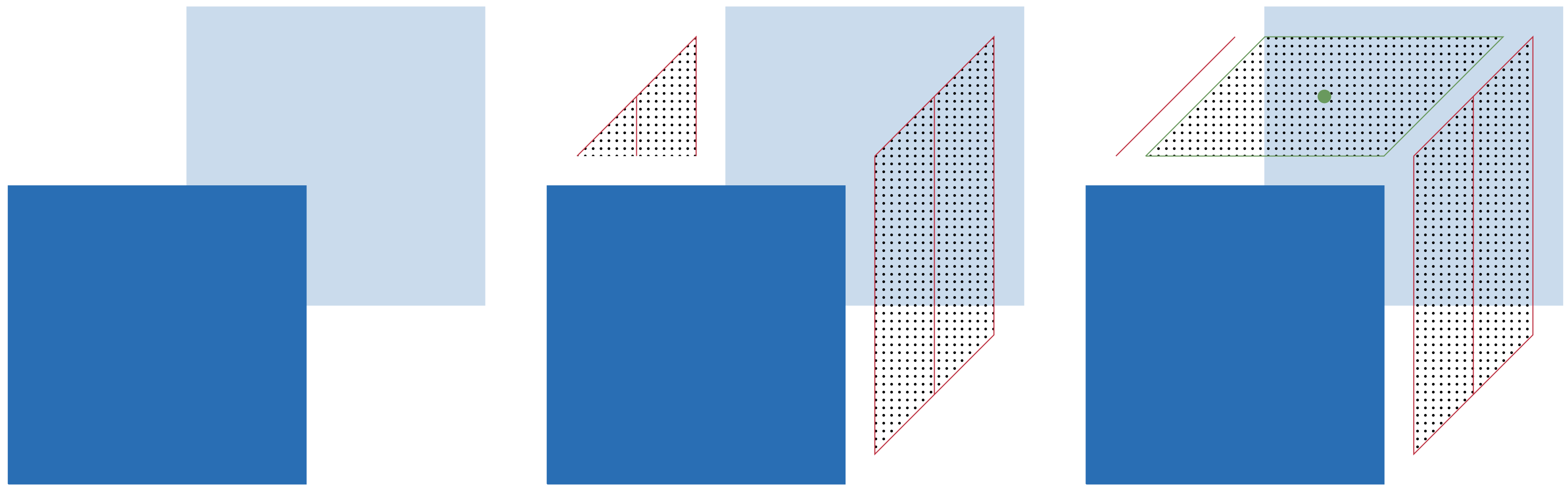
$\nu\mathsf{Set}_m$		$:$	HSet_{m+1}
$\nu\mathsf{Set}_m$		$\triangleq$	$\nu\mathsf{Set}_m^{\geq 0}(*)$
$\nu\mathsf{Set}_m^{\geq n}$	$(D : \nu\mathsf{Set}_m^{<n})$	$:$	HSet_{m+1}
$\nu\mathsf{Set}_m^{\geq n}$	D	$\triangleq$	$\Sigma R : \nu\mathsf{Set}_m^{=n}(D).\nu\mathsf{Set}_m^{\geq n+1}(D,R)$

Truncated ν-sets

$\nu\mathsf{Set}_m^{<n}$		$:$	HSet_{m+1}
$\nu\mathsf{Set}_m^{<0}$		$\triangleq$	unit
$\nu\mathsf{Set}_m^{<n'+1}$		$\triangleq$	$\Sigma D : \nu\mathsf{Set}_m^{<n'}.\nu\mathsf{Set}_m^{=n'}(D)$
$\nu\mathsf{Set}_m^{=n}$	$(D : \nu\mathsf{Set}_m^{<n})$	$:$	HSet_{m+1}
$\nu\mathsf{Set}_m^{=n}$	D	$\triangleq$	$\mathsf{fullframe}_m^n(D) \to \mathsf{HSet}_m$

where $\mathsf{fullframe}_m^n$ is defined by mutual recursive construction (see next slides)

The recursive process used to build frames from layers of paintings



The recursive construction, formally

$\mathsf{fullframe}^{n}$	$(D : \nu\mathsf{Set}_{m}^{\leq n})$	$:$	HSet_{m}
$\mathsf{fullframe}^{n}$	D	$\triangleq$	$\mathsf{frame}^{n,n}(D)$
$\mathsf{frame}^{n,p,p\leq n}$	$(D : \nu\mathsf{Set}_{m}^{\leq n})$	$:$	HSet_{m}
$\mathsf{frame}^{n,0}$	D	$\triangleq$	unit
$\mathsf{frame}^{n,p'+1}$	D	$\triangleq$	$\Sigma d : \mathsf{frame}^{n,p'}(D).\ \mathsf{layer}^{n,p'}(d)$
$\mathsf{layer}^{n,p,p<n}$	$\{D : \nu\mathsf{Set}_{m}^{\leq n}\}$ $(d : \mathsf{frame}^{n,p}(D))$	$:$	HSet_{m}
$\mathsf{layer}^{n,p}$	$D\ d$	$\triangleq$	$\Pi\omega.\mathsf{painting}^{n-1,p}(D.2)(\mathsf{restr}^{n,p}_{\mathsf{frame},\omega,p}(d))$
$\mathsf{painting}^{n,p,p\leq n}$	$(D : \nu\mathsf{Set}_{m}^{\leq n})$ $(E : \nu\mathsf{Set}_{m}^{=n}(D))$ $(d : \mathsf{frame}^{n,p}(D))$	$:$	HSet_{m}
$\mathsf{painting}^{n,p,p=n}$	$D\ E\ d$	$\triangleq$	$E(d)$
$\mathsf{painting}^{n,p,p<n}$	$D\ E\ d$	$\triangleq$	$\Sigma l : \mathsf{layer}^{n,p}(d).\ \mathsf{painting}^{n,p+1}(E)(d,l)$

where we need to define $\mathsf{restr}_{\mathsf{frame}}$ (see next slide)

The recursive construction: restrictions ("faces")

$\mathsf{restr}^{n,p,p\le q\le n-1}_{\mathsf{frame},\epsilon,q}$	$\{D:\nu\mathsf{Set}^{<n}\}$ $(d:\mathsf{frame}^{n,p}(D))$	$:$	$\mathsf{frame}^{n-1,p}(D.1)$
$\mathsf{restr}^{n,0}_{\mathsf{frame},\epsilon,q}$	$D\ *$	$\triangleq$	$*$
$\mathsf{restr}^{n,p'+1}_{\mathsf{frame},\epsilon,q}$	$D\ (d,l)$	$\triangleq$	$(\mathsf{restr}^{n,p'}_{\mathsf{frame},\epsilon,q}(d),\mathsf{restr}^{n,p'}_{\mathsf{layer},\epsilon,q-1}(l))$
$\mathsf{restr}^{n,p,p\le q\le n-2}_{\mathsf{layer},\epsilon,q}$	$\{D:\nu\mathsf{Set}^{<n}\}$ $\{d:\mathsf{frame}^{n,p}(D)\}$ $(l:\mathsf{layer}^{n,p}(d))$	$:$	$\mathsf{layer}^{n-1,p}(\mathsf{restr}^{n,p}_{\mathsf{frame},\epsilon,q+1}(d))$
$\mathsf{restr}^{n,p}_{\mathsf{layer},\epsilon,q}$	$D\ d\ l$	$\triangleq$	$\lambda\omega.(\overrightarrow{\mathsf{coh}^{n,p}_{\mathsf{frame},\epsilon,\omega,q,p}(d)(\mathsf{restr}}^{n-1,p}_{\mathsf{painting},\epsilon,q}(D.2)(l_\omega)))$
$\mathsf{restr}^{n,p,p\le q\le n-1}_{\mathsf{painting},\epsilon,q}$	$(D:\nu\mathsf{Set}^{<n})$ $(E:\nu\mathsf{Set}^{=n}(D))$ $(d:\mathsf{frame}^{n,p}(D))$ $(c:\mathsf{painting}^{n,p}(E)(d))$	$:$	$\mathsf{painting}^{n-1,p}(D.2)(\mathsf{restr}^{n,p}_{\mathsf{frame},\epsilon,q}(d))$
$\mathsf{restr}^{n,p,p=q}_{\mathsf{painting},\epsilon,q}$	$D\ E\ d\ (l,\_)$	$\triangleq$	l_ϵ
$\mathsf{restr}^{n,p,p<q}_{\mathsf{painting},\epsilon,q}$	$D\ E\ d\ (l,c)$	$\triangleq$	$(\mathsf{restr}^{n,p}_{\mathsf{layer},\epsilon,q-1}(l),\mathsf{restr}^{n,p+1}_{\mathsf{painting},\epsilon,q}(E)(c))$

where we need to define $\mathsf{coh}_{\mathsf{frame}}$ (see next slide)

The recursive construction: coherences

$\mathsf{coh}^{n,p,p\le r\le q\le n-2}_{\mathsf{frame},\varepsilon,\omega,q,r}$	$\{D:\nu\mathsf{Set}^{<n}\}$ $(d:\mathsf{frame}(D))$	$:$	$\mathsf{restr}^{n-1,p}_{\mathsf{frame},\varepsilon,q}(\mathsf{restr}^{n,p}_{\mathsf{frame},\omega,r}(d))$ $= \mathsf{restr}^{n-1,p}_{\mathsf{frame},\omega,r}(\mathsf{restr}^{n,p}_{\mathsf{frame},\varepsilon,q+1}(d))$
$\mathsf{coh}^{n,0}_{\mathsf{frame},\varepsilon,\omega,q,r}$	$D\ *$	$\triangleq$	$\mathsf{refl}(*)$
$\mathsf{coh}^{n,p'+1}_{\mathsf{frame},\varepsilon,\omega,q,r}$	$D\ (d,l)$	$\triangleq$	$(\mathsf{coh}^{n,p'}_{\mathsf{frame},\varepsilon,\omega,q,r}(d), \mathsf{coh}^{n,p'}_{\mathsf{layer},\varepsilon,\omega,q-1,r-1}(l))$
$\mathsf{coh}^{n,p,p\le r\le q\le n-3}_{\mathsf{layer},\varepsilon,\omega,q,r}$	$\{D:\nu\mathsf{Set}^{<n}\}$ $\{d:\mathsf{frame}(D)\}$ $(l:\mathsf{layer}(d))$	$:$	$\mathsf{restr}^{n-1,p}_{\mathsf{layer},\varepsilon,q}(\mathsf{restr}^{n,p}_{\mathsf{layer},\omega,r}(l))$ $= \mathsf{restr}^{n-1,p}_{\mathsf{layer},\omega,r}(\mathsf{restr}^{n,p}_{\mathsf{layer},\varepsilon,q+1}(l))$
$\mathsf{coh}^{n,p}_{\mathsf{layer},\varepsilon,\omega,q,r}$	$D\ d\ l$	$\triangleq$	$\lambda\theta.\ \mathsf{coh}^{n-1,p}_{\mathsf{painting},\varepsilon,\omega,q,r}(D.2)(l_\theta)$
$\mathsf{coh}^{n,p,p\le r\le q\le n-2}_{\mathsf{painting},\varepsilon,\omega,q,r}$	$\{D:\nu\mathsf{Set}^{<n}\}$ $(E:\nu\mathsf{Set}^{=n}(D))$ $\{d:\mathsf{frame}(D)\}$ $(c:\mathsf{painting}(E)(d))$	$:$	$\mathsf{restr}^{n-1,p}_{\mathsf{painting},\varepsilon,q}(D.2)(\mathsf{restr}^{n,p}_{\mathsf{painting},\omega,r}(E)(c))$ $= \mathsf{restr}^{n-1,p}_{\mathsf{painting},\omega,r}(D.2)(\mathsf{restr}^{n,p}_{\mathsf{painting},\varepsilon,q+1}(E)(c))$
$\mathsf{coh}^{n,p,p=r}_{\mathsf{painting},\varepsilon,\omega,q,r}$	$D\ E\ d\ (l,\_)$	$\triangleq$	$\mathsf{refl}(\mathsf{restr}^{n-1,p}_{\mathsf{painting},\varepsilon,q}(D.2)(l_\omega))$
$\mathsf{coh}^{n,p,p<r}_{\mathsf{painting},\varepsilon,\omega,q,r}$	$D\ E\ d\ (l,c)$	$\triangleq$	$(\mathsf{coh}^{n,p}_{\mathsf{layer},\varepsilon,\omega,q-1,r-1}(l), \mathsf{coh}^{n,p+1}_{\mathsf{painting},\varepsilon,\omega,q,r}(E)(c))$

where we hide many steps of equational reasoning: proof-irrelevance of equality in **HSet**, identification of equality of pairs and pairs of equalities, groupoid properties of equality

About the formalisation

Complex proof of termination

- made several unsuccessful attempts
- construction completed in Rocq in Apr 2022
  (inductively building 3 levels at once with two subinductions)
- degeneracies completed in Nov 2024
- we are working on a simplification saving a lot of equational reasoning
- code at `https://github.com/artagnon/bonak`

Note: “paper” construction also fully formulated in Agda (w/o termination)

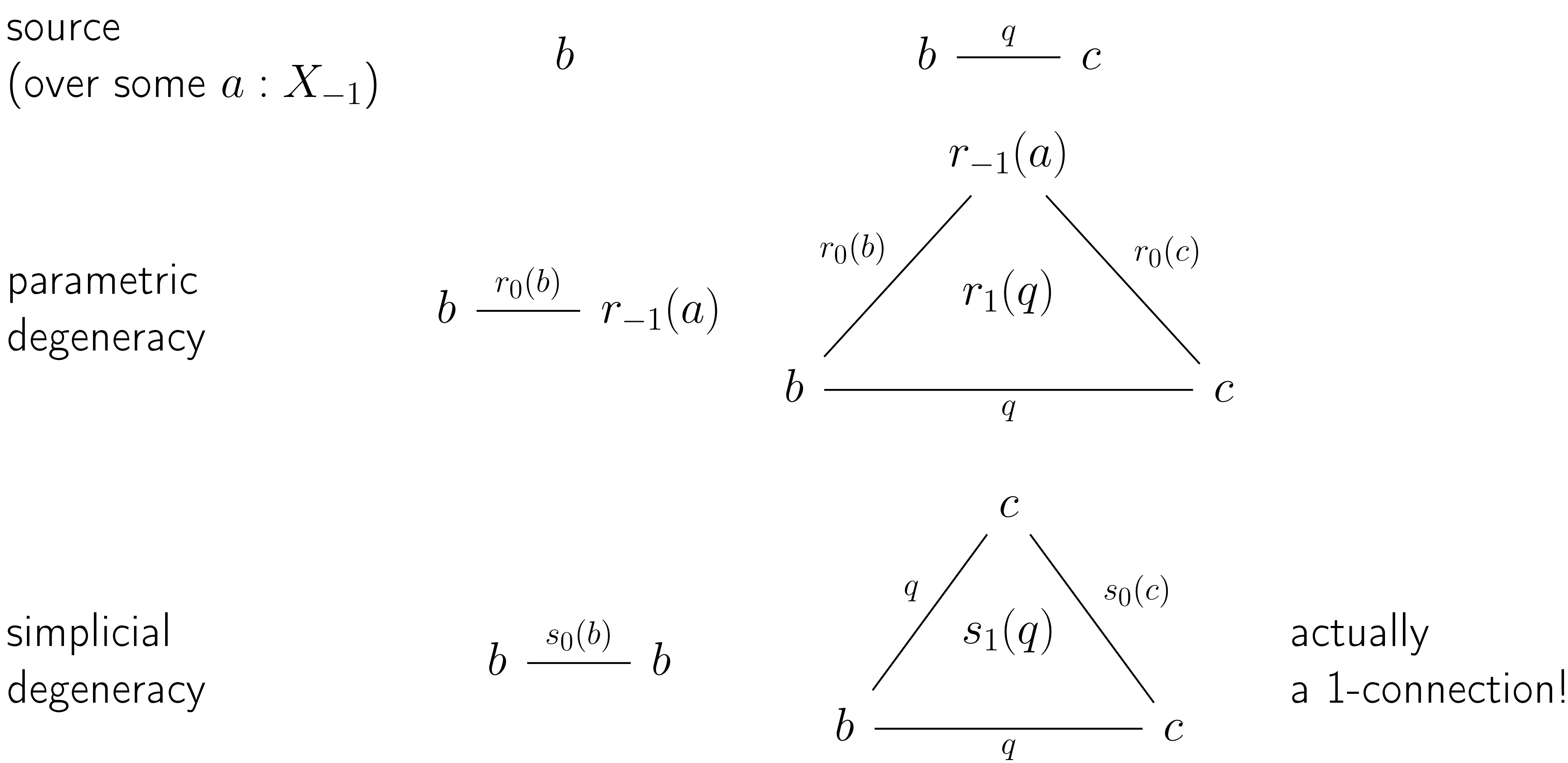
Adding (one) degeneracy (in the last direction)

fibred form $\quad vs \quad$ *indexed form*

Y_0 : HSet	X_0 : HSet	(points)
$\uparrow\uparrow\downarrow$		
Y_1 : HSet	$X_1 : X_0 \times X_0 \to \mathsf{HSet}$	(segments)
$\uparrow\uparrow\uparrow\uparrow\downarrow$	$r_0 : \Pi x_0 : X_0.\, X_1(x_0, x_0)$	
Y_2 : HSet	$X_2 : \Pi(x^0_{LL}, x^0_{LR}).\, \Pi x^1_{L*} : X_1(x^0_{LL}, x^0_{LR}).$ $\Pi(x^0_{RL}, x^0_{RR}).\, \Pi x^1_{R*} : X_1(x^0_{RL}, x^0_{RR}).$ $X_1(x^0_{LL}, x^0_{RL}) \times X_1(x^0_{LR}, x^0_{RR}) \to \mathsf{HSet}$	(squares)
+ coherences	$r_1 : \Pi(x^0_L, x^0_R) : (X_0 \times X_0).\, \Pi x^1 : X_1(x^0_L, x^0_R).$ $X_2((x^0_L, x^0_L), r_0(x^0_L), (x^0_R, x^0_R), r_0(x^0_R), (x^1, x^1))$	
$\vdots$	$\vdots$	

The added degeneracy is *parametric*: in the binary case, it gives a standard cubical degeneracy; in the unary case, it gives a ParamTT-like degeneracy and *not* a simplicial degeneracy

First, our degeneracy implies a distinguished point $r_{-1}(a)$ for any $a : X_{-1}$. Then:



Adding a degeneracy

For any $(X_0, X_1, ...) : \nu\mathsf{Set}$, we define a stream of degeneracies:

$$
\begin{array}{l}
\nu\mathsf{reflSet}(X_0, X_1, ...) \quad \triangleq \\
\quad \Sigma r_0 : \Pi d : \mathsf{fullframe}^0 . \Pi x : X_0(d). X_1(\mathsf{refl}^0_{\mathsf{fullframe}}(d), \lambda\epsilon.\, x). \\
\quad \Sigma r_1 : \Pi d : \mathsf{fullframe}^1(X_0). \Pi x : X_1(d). X_2(\mathsf{refl}^1_{\mathsf{fullframe}}(r_0)(d), \lambda\epsilon.\, x). \\
\quad \Sigma r_2 : \Pi d : \mathsf{fullframe}^2(X_0, X_1). \Pi x : X_2(d). X_3(\mathsf{refl}^2_{\mathsf{fullframe}}(r_0, r_1)(d), \lambda\epsilon.\, x). \\
\quad \ldots
\end{array}
$$

where

$$\mathsf{refl}^n_{\mathsf{fullframe}}(r_{-1}, ..., r_{n-1}) : \mathsf{fullframe}^n(X_{-1}, ..., X_{n-1}) \to \mathsf{frame}^{n+1,n}(X_{-1}, ..., X_n)$$

computes the n first layers of the border of $r_n(d)(x)$, knowing that the last layer is made of ν times x itself, so that

$$(\mathsf{refl}^n_{\mathsf{fullframe}}(r_{-1}, ..., r_{n-1})(d), \lambda\epsilon.\, x) : \mathsf{frame}^{n+1,n+1}(X_{-1}, ..., X_n)$$

is a full frame.

Adding a degeneracy

On the way, we need two coherence conditions:

$$\mathsf{idrestrrefl}^{n}_{\mathsf{frame},\epsilon}(r_{-1}, ..., r_{n-1})\ (d : \mathsf{fullframe}^{n}(X_0, ..., X_{n-1})) :$$
$$\mathsf{restr}^{n,n}_{\mathsf{frame},\epsilon,n}(\mathsf{refl}^{n}_{\mathsf{fullframe}}(r_{-1}, ..., r_{n-1})(d)) = d$$

$$\mathsf{cohrestrrefl}^{n}_{\mathsf{frame},\epsilon,p<n}(r_{-1}, ..., r_{n-1})\ (d : \mathsf{frame}^{n,p}(X_0, ..., X_{n-1})) :$$
$$\mathsf{restr}^{n,p}_{\mathsf{frame},\epsilon,p}(\mathsf{refl}^{n,p}_{\mathsf{frame}}(r_{-1}, ..., r_{n-1})(d)) = \mathsf{refl}^{n-1,p}_{\mathsf{frame}}(r_{-1}, ..., r_{n-2})(\mathsf{restr}^{n-1,p}_{\mathsf{frame},\epsilon,p}(d))$$

where $\mathsf{refl}^{n,p}_{\mathsf{frame}}$ generalises $\mathsf{refl}^{n}_{\mathsf{fullframe}}$ to prefixes of $\mathsf{fullframe}^{n}$:

$$\mathsf{refl}^{n,p}_{\mathsf{frame}}(r_{-1}, ..., r_{n-1}) : \mathsf{frame}^{n,p}(X_{-1}, ..., X_{n-1}) \to \mathsf{frame}^{n+1,p}(X_{-1}, ..., X_{n})$$

Summary

- Machine-checked parametricity-based definition of *indexed* presheaves
- Uniformly represents simplicial and cubical sets
- Addition of one (parametric) degeneracy in the last direction completed
- More compact definition in progress, relying on finer-grain dependencies between the different components of the construction