An existential-free theory of arithmetic in all finite types*

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Gödel's Consistency Proof of PA†

1. In Gödel 1933, he gave a translation, which is called the "negative translation" nowadays, of formal proofs of PA into formal proofs of HA.
2. In Gödel 1958, he gave a translation, which is called the "Dialectica (or functional) interpretation" nowadays, of formal proofs of HA into formal proofs of his quantifier-free theory T in all finite types:
   - $\mathbb{N}$ is a type;
   - If σ, τ are types, then $\sigma \to \tau$ is a type.

By these two steps, the consistency of PA is finitistically reducible to the consistency of T.

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- K. Gödel, Zur intuitionistischen Arithmetik und Zahlentheorie. Ergebnisse eines mathematischen Kolloquiums, 4:34–38, 1933.
- K. Gödel, Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica, 12:280–287, 1958.

If one employs classical and intuitionistic arithmetic in all finite types, namely PA^ω and HA^ω respectively, Gödel's achievements can be mentioned as follows:

1. The consistency of PA^ω is finitistically reducible to the consistency of HA^ω.
2. The consistency of HA^ω is finitistically reducible to the consistency of T.

Axioms and Rules of E-HA$^\omega$ (cf. Gödel 1958)

- Axioms of contraction: $A \vee A \to A, A \to A \wedge A$;
- Axioms of weakening: $A \to A \vee B, A \wedge B \to A$;
- Axioms of permutation: $A \vee B \to B \vee A, A \wedge B \to B \wedge A$;
- Ex falso quodlibet: $\perp \to A$;
- Quantifier axioms: $\forall x A \to A[t/x], A[t/x] \to \exists x A$;
- Equality axioms for $=_{\mathbb{N}}$;
- Higher type extensionality axiom[‡]:

$$E_{\rho,\tau} : \forall z^{\rho\to\tau}, x^\rho, y^\rho \left(x =_\rho y \to zx =_\tau zy\right);$$

- Induction axiom;
- Defining axioms for combinators and recursor:

$$\begin{aligned}
(\Pi) :&\quad \Pi_{\rho,\tau} x^\rho y^\tau =_\rho x^\rho; \\
(\Sigma) :&\quad \Sigma_{\delta,\rho,\tau} xyz =_\tau xz(yz); \\
(R) :&\quad \begin{cases} R_\rho 0yz =_\rho y; \\ R_\rho(Sx)yz =_\rho z(R_\rho xyz)x. \end{cases}
\end{aligned}$$

[‡]For $\rho := \rho_1 \to \cdots \to \rho_k \to \mathbb{N}$, $s =_\rho t$ is $\forall y_1^{\rho_1}, \ldots, y_k^{\rho_k}(sy_1 \ldots y_k =_{\mathbb{N}} ty_1 \ldots y_k)$.

- Modus ponens and syllogism: $\dfrac{A, \quad A \to B}{B}$, $\dfrac{A \to B, \quad B \to C}{A \to C}$;
- Exportation and importation: $\dfrac{A \wedge B \to C}{A \to (B \to C)}$, $\dfrac{A \to (B \to C)}{A \wedge B \to C}$;
- Expansion: $\dfrac{A \to B}{C \vee A \to C \vee B}$;
- Quantifier rules: $\dfrac{B \to A}{B \to \forall x A}$, $\dfrac{A \to B}{\exists x A \to B}$ for $x \notin \mathrm{FV}(B)$.

Remark. The terms of E-HA$^\omega$ are the same as those of T.

Definition

- HA$^\omega$ is obtained from E-HA$^\omega$ by restricting the extensionality axiom appropriately.
- E-PA$^\omega$ and PA$^\omega$ are E-HA$^\omega$ $+ \mathrm{LEM}$ and HA$^\omega$ $+ \mathrm{LEM}$ respectively.

- In connection with constructivism, many kinds of realizability interpretation have been studied.
- In particular, Kreisel's modified (generalized) realizability interpretation (cf. Kreisel 1959, 1962) is a sort of direct formalization of the BHK-notion of constructive proofs in the language of arithmetic in all finite types.
- Now we consider a $\exists$-free (containing neither $\exists$ nor $\vee$) fragment $\text{E-HA}^\omega_{\text{ef}}$ of E-HA^ω. In fact, Gödel's T can be seen as a subtheory of $\text{E-HA}^\omega_{\text{ef}}$, and hence,

$$T \subset \text{E-HA}^\omega_{\text{ef}} \subset \text{E-HA}^\omega.$$

- $\text{E-HA}^\omega_{\text{ef}}$ is similar to Kreisel's $\exists$-free theory $\text{HA}^\omega_{\text{NF}}$ in Kreisel 1962 for the verification of the soundness of the modified realizability interpretation. On the other hand, our theory $\text{E-HA}^\omega_{\text{ef}}$ is consistent with classical logic in contrast to that $\text{HA}^\omega_{\text{NF}}$ contains some continuity axiom which is inconsistent with classical logic.

E-HA$^\omega_{\mathrm{ef}}$

- The type structure and the language of E-HA$^\omega_{\mathrm{ef}}$ are the same as those for E-HA$^\omega$ except that E-HA$^\omega_{\mathrm{ef}}$ has only $\wedge, \rightarrow$ and $\forall x^\rho$ (for any type ρ) as logical connectives.
- The terms of E-HA$^\omega_{\mathrm{ef}}$ are the same as those of E-HA$^\omega$ (i.e. those of T).
- Axioms and rules of E-HA$^\omega_{\mathrm{ef}}$ consists of the axioms and rules of E-HA$^\omega$ which contain neither $\exists$ nor $\vee$.

Axioms of E-$\mathrm{HA}^{\omega}_{\mathrm{ef}}$

- Axioms of contraction: $A \vee A \to A, A \to A \wedge A$;
- Axioms of weakening: ~~$A \to A \vee B$~~, $A \wedge B \to A(a)$;
- Axioms of permutation: ~~$A \vee B \to B \vee A$~~, $A \wedge B \to B \wedge A$;
- Ex falso quodlibet: $\perp \to A$;
- Quantifier axioms: $\forall x A \to A[t/x]$, ~~$A[t/x] \to \exists x A$~~;
- Equality axioms for $=_{\mathbb{N}}$;
- Higher type extensionality axiom:

$$E_{\rho,\tau} : \forall z^{\rho\to\tau}, x^{\rho}, y^{\rho} \left(x =_{\rho} y \to zx =_{\tau} zy\right);$$

- Induction axiom;
- Defining axioms for combinators and recursor:

$$\begin{aligned} (\Pi) &: \quad \Pi_{\rho,\tau} x^{\rho} y^{\tau} =_{\rho} x^{\rho}; \\ (\Sigma) &: \quad \Sigma_{\delta,\rho,\tau} xyz =_{\tau} xz(yz); \\ (R) &: \quad \begin{cases} R_{\rho} 0yz =_{\rho} y; \\ R_{\rho}(Sx)yz =_{\rho} z(R_{\rho}xyz)x. \end{cases} \end{aligned}$$

Rules of E-$\mathrm{HA}^{\omega}_{\mathrm{ef}}$

- Modus ponens and syllogism:
  $\dfrac{A, \quad A \to B}{B}, \quad \dfrac{A \to B, \quad B \to C}{A \to C};$
- Exportation and importation:
  $\dfrac{A \wedge B \to C}{A \to (B \to C)}, \quad \dfrac{A \to (B \to C)}{A \wedge B \to C};$
- ~~Expansion: $\dfrac{A \to B}{C \vee A \to C \vee B}$~~;
- Quantifier rules: $\dfrac{B \to A}{B \to \forall x A}$, ~~$\dfrac{A \to B}{\exists x A \to B}$~~ for $x \notin \mathrm{FV}(B)$.

Definition (Gödel-Gentzen Negative Translation)

- $A^N :\equiv \neg\neg A$ for prime A;
- $(A \wedge B)^N :\equiv A^N \wedge B^N$;
- $(A \vee B)^N :\equiv \neg\left(\neg A^N \wedge \neg B^N\right)$;
- $(A \to B)^N :\equiv A^N \to B^N$;
- $(\forall x^\rho A)^N :\equiv \forall x^\rho A^N$;
- $(\exists x^\rho A)^N :\equiv \neg\forall x^\rho \neg A^N$.

Remark

- A^N is always $\exists$-free (contains neither $\exists$ nor $\vee$).
- For $\exists$-free A, A^N is equivalent to A itself (over $\text{E-HA}^\omega_{\text{ef}}$).

Theorem

If $\text{E-PA}^\omega + \Delta \vdash A$, *then* $\text{E-HA}^\omega_{\text{ef}} + \Delta^N \vdash A^N$.

Modified Realizability

The modified (generalized) realizability interpretation[§], which is a sort of intuitionistic semantics (in the sense of BHK) of finite-type arithmetic, can be seen as a variant of the Dialectica interpretation.

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§

- G. Kreisel. Interpretation of analysis by means of constructive functionals of finite types, In Constructivity in mathematics, Proceedings of the colloquium held at Amsterdam 1957, pp. 101–128, 1959.
- G. Kreisel, On weak completeness of intuitionistic predicate logic, Journal of Symbolic Logic 27, pp.139–158, 1962.

Definition (Modified Realizability)

For prime A, $A^{mr} :\equiv \exists \underline{w}\,(\underline{w} \text{ mr } A) :\equiv A$ with $\underline{w}$ being empty.
Let $A^{mr} :\equiv \exists \underline{x}\,(\underline{x} \text{ mr } A)$, $B^{mr} :\equiv \exists \underline{y}\,(\underline{y} \text{ mr } B)$. Then,

- $(A \wedge B)^{mr} :\equiv \exists \underline{x}, \underline{y}\,(\underline{x}, \underline{y} \text{ mr } (A \wedge B))$
  $:\equiv \exists \underline{x}, \underline{y}\,(\underline{x} \text{ mr } A \wedge \underline{y} \text{ mr } B)$;
- $(A \vee B)^{mr} :\equiv \exists w^{\mathbb{N}}, \underline{x}, \underline{y}(z, \underline{x}, \underline{y} \text{ mr } (A \vee B))$
  $:\equiv \exists w, \underline{x}, \underline{y}\left(\left(w =_{\mathbb{N}} 0 \to \underline{x} \text{ mr } A\right) \wedge \left(w \neq 0 \to \underline{y} \text{ mr } B\right)\right)$;
- $(A \to B)^{mr} :\equiv \exists \underline{w}\,(\underline{w} \text{ mr } (A \to B))$
  $:\equiv \exists \underline{w} \forall \underline{x}\left(\underline{x} \text{ mr } A \to \underline{w}\,\underline{x} \text{ mr } B\right)$;
- $(\forall z^{\rho} A)^{mr} :\equiv \exists \underline{w}\,(\underline{w} \text{ mr } \forall z A) :\equiv \exists \underline{w} \forall z(\underline{w}\, z \text{ mr } A)$;
- $(\exists z^{\rho} A)^{mr} :\equiv \exists z, \underline{x}\,(z, \underline{x} \text{ mr } \exists z A) :\equiv \exists z, \underline{x}\,(\underline{x} \text{ mr } A)$.

Here $\underline{x}, \underline{y}$ are tuples of distinct variables, $\underline{w}$ is a tuple of flesh variables whose length and types are determined by the logical structure of the formula in question, and $\underline{w}\,\underline{x}$ denotes $w_1\underline{x}, \ldots, w_n\underline{x}$ where $w_i\underline{x}$ denotes $w_i x_1, \ldots x_k$ for tuples $\underline{w} := w_1, \ldots, w_n$ and $\underline{x} := x_1 \ldots, x_k$ of suitable types.

Remark

1. For $A^{mr} :\equiv \exists \underline{x}\,(\underline{x} \text{ mr } A)$, $(\underline{x} \text{ mr } A)$ is $\exists$-free.
2. If A is $\exists$-free, $A^{mr} \equiv A$.

Theorem (Soundness of the modified realizability)

If $\text{E-HA}^\omega + \text{AC}^\omega + \text{IP}^\omega_{\text{ef}} + \Delta_{\text{ef}} \vdash A$*, then there exists a tuple of terms* $\underline{t}$ *of* T *such that* $\text{E-HA}^\omega_{\text{ef}} + \Delta_{\text{ef}} \vdash \underline{t} \text{ mr } A$ *and all the variables in* $\underline{t}$ *are in* $\text{FV}(A)$*.*

- $\text{AC}^{\rho,\tau} : \forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\rho\to\tau} \forall x^\rho A(x, Yx)$;
- $\text{IP}^\rho_{\text{ef}} : (A_{\text{ef}} \to \exists x^\rho B(x)) \to \exists x^\rho (A_{\text{ef}} \to B(x))$.

Discussion on Gödel's Consistency Proofs

- Both of the negative translation and the modified realizability interpretation (finitistically) reduce the consistency of HA to that of $\text{E-HA}^{\omega}_{\text{ef}}$. In addition, both of them do not change $\text{E-HA}^{\omega}_{\text{ef}}$-formulas anymore.
- Since $\text{E-HA}^{\omega}_{\text{ef}}$ is self-closed theory for the soundness of the modified realizability interpretation, $\text{E-HA}^{\omega}_{\text{ef}}$ can be regarded as a constructive foundational base. On the other hand, T is a finitistic base in an extended sense.
- From this perspective, one may argue that Gödel firstly showed in Gödel 1933 the consistency of PA based on a constructive foundation, and secondly showed in Gödel 1958 the same thing based on a finitistic foundation (in an extended sense).

Spector's Consistency Proof of $\mathsf{PA}^\omega + \mathrm{AC}^{\mathbb{N},\mathbb{N}}$

In his posthumous paper, Spector 1962 introduced the notion of bar recursion and extend Gödel's consistency proof of PA to that of classical analysis $\mathsf{PA}^\omega + \mathrm{AC}^{\mathbb{N},\mathbb{N}}$ as follows:

$$\mathsf{PA}^\omega + \mathrm{AC}^{\mathbb{N},\mathbb{N}} \vdash \bot \Longrightarrow \mathsf{HA}^\omega + \mathrm{AC}^{\mathbb{N},\mathbb{N}} + \mathrm{DNS}^{\mathbb{N}} \vdash \bot \Longrightarrow T + \mathrm{BR}^{\mathbb{N}} \vdash \bot.$$

- $\mathrm{DNS}^\tau : \forall x^\tau \neg\neg A(x) \to \neg\neg\forall x^\tau \; A(x)$,
- The defining axiom of $\mathrm{BR}^{\mathbb{N}}$:

$$\begin{cases} Y\hat{s} < |s| & \to & \mathbf{B}YGHs =_\tau Gs, \\ Y\hat{s} \geq |s| & \to & \mathbf{B}YGHs =_\tau H(\lambda w^{\mathbb{N}}.\,\mathbf{B}YGH(s * \langle w \rangle))s, \end{cases}$$

where s is a finite sequence of objects of type $\mathbb{N}$.

Definition

Let $\mathrm{N\text{-}AC}^{\omega}_{\mathrm{ef}}$ consist of

$$\mathrm{N\text{-}AC}^{\sigma,\tau}_{\mathrm{ef}} : \forall x^{\sigma} \neg \forall y^{\tau} \neg A_{\mathrm{ef}}(x, y) \to \neg \forall Y^{\sigma \to \tau} \neg \forall x^{\sigma} A_{\mathrm{ef}}(x, Yx).$$

Remark

$\mathrm{N\text{-}AC}^{\omega}_{\mathrm{ef}}$ is $\exists$-free, and hence, it is not changed anymore by the negative translation and the modified realizability.

Theorem

If E-HA$^{\omega}$ $+ \mathrm{AC}^{\omega} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \mathrm{DNS}^{\omega} \vdash A$, *then there exists a tuple of terms* $\underline{t}$ *of* E-HA$^{\omega}_{\mathrm{ef}}$ *such that* E-HA$^{\omega}_{\mathrm{ef}}$ $+ \mathrm{N\text{-}AC}^{\omega}_{\mathrm{ef}} \vdash \underline{t} \ \mathrm{mr}\ A$ *and all the variables in* $\underline{t}$ *are in* $\mathrm{FV}(A)$.

Theorem

For any instance A of $\mathrm{MBI}^{\mathbb{N}}$ *(Brouwer's bar theorem), there exists a tuple of terms* $\underline{t}$ *of* $T + \mathrm{BR}^{\mathbb{N}}$ *such that*

$$\textsf{E-HA}^{\omega}_{\textsf{ef}} + \mathrm{N\text{-}AC}^{\mathbb{N},\mathbb{N}}_{\mathrm{ef}} + \mathrm{BR}^{\mathbb{N}} \vdash \underline{t} \ \mathrm{mr} \ A$$

and all the variables in $\underline{t}$ *are in* $\mathrm{FV}(A)$.

Remark. $\textsf{E-HA}^{\omega}_{\textsf{ef}} + \mathrm{N\text{-}AC}^{\mathbb{N},\mathbb{N}}_{\mathrm{ef}} + \mathrm{BR}^{\mathbb{N}}$ is consistent (relative to $T + \mathrm{BR}^{\mathbb{N}}$) but $\textsf{E-HA}^{\omega}_{\textsf{ef}} + \mathrm{N\text{-}AC}^{\mathbb{N}\to\mathbb{N},\mathbb{N}}_{\mathrm{ef}} + \mathrm{BR}^{\mathbb{N}}$ is already inconsistent as $\mathrm{BR}^{\mathbb{N}}$ conflicts with $\Pi^0_1\text{-}\mathrm{AC}^{\mathbb{N}\to\mathbb{N},\mathbb{N}}$ classically.

Conclusion

E-$\mathrm{HA}^{\omega}_{\mathrm{ef}}$, E-$\mathrm{HA}^{\omega}_{\mathrm{ef}} + \mathrm{N}\text{-}\mathrm{AC}^{\mathbb{N},\mathbb{N}}_{\mathrm{ef}}$ and E-$\mathrm{HA}^{\omega}_{\mathrm{ef}} + \mathrm{N}\text{-}\mathrm{AC}^{\omega}_{\mathrm{ef}}$ seem to be robust theories of neutral constructivism corresponding to E-PA^{ω}, E-$\mathrm{PA}^{\omega} + \mathrm{AC}^{\mathbb{N},\mathbb{N}}$ and E-$\mathrm{PA}^{\omega} + \mathrm{AC}^{\omega}$ respectively.

Questions

1. How is the relation between these theories and type theories?
2. What amount of mathematics can be developed in these systems?
3. What can we say about the relation between the negative translation and the modified realizability interpretation in more general context?