# Examples and counter-examples of injective types in univalent mathematics

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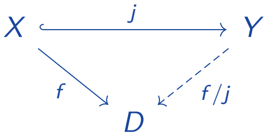
# Motivation

- We work in univalent foundations a.k.a. homotopy type theory (HoTT).
- Injective types were used by Escardó to construct infinite searchable types, see his *TYPES 2019* abstract, but the topic has a rich theory of its own.
- In this talk, we present new examples and counter-examples of injective types.

# Injective types

- Def. A type $D$ is (algebraically) injective if for every *embedding* $j : X \hookrightarrow Y$, any map $f : X \to D$ into $D$ has a designated extension $f/j$.

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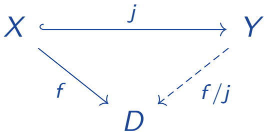


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- Recall: **embedding** $\approx$ homotopically well-behaved injection.
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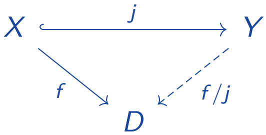


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- The notion of injectivity is sensitive to universe levels, so we really study $\mathcal{U}, \mathcal{V}$-injective types where $X : \mathcal{U}$ and $Y : \mathcal{V}$, but we largely ignore this in this talk.

# Examples of injective types

- Any univalent universe $\mathcal{U}$
- The type $\Omega_{\mathcal{U}}$ of propositions in a universe $\mathcal{U}$
- The type $\mathcal{L}\,X \coloneqq \Sigma(P : \Omega\mathcal{U}), (P \to X)$ of partial elements of a type $X : \mathcal{U}$
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New examples

- The type of iterative (multi)sets in $\mathcal{U}$
- The types of small $\infty$-magmas, monoids and groups
- The underlying set of any sup-compete poset, or more generally, of any pointed dcpo

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- For **subtypes** there is a necessary and sufficient criterion:
  Thm. A subtype $\Sigma(d : D), P\, d$ of an injective type $D$ is injective if and only if we have $f : D \to D$ such that for all $d : D$
  (i) $P$ holds for $f\, d$ and
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- Ex. The injectivity of $\Omega_{\mathcal{U}}$ follows by taking $P :=$ is-prop and $f$ to be the propositional truncation.
  This generalizes to any **reflective subuniverse**.

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- Thm. If there is a $\mathcal{U},\mathcal{U}$-injective type in $\mathcal{U}$ with two distinct points, then the type $\Omega_{\neg\neg} := \Sigma(P : \Omega_{\mathcal{U}}) \times (\neg\neg P \to P)$ of $\neg\neg$-stable propositions in $\mathcal{U}$, whose native universe is $\mathcal{U}^+$, is equivalent to a type in $\mathcal{U}$.

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- This theorem is comparable to a result of Aczel et al.: in the predicative set theory **CZF** it is consistent that the only injective *sets* (as opposed to *classes*) are singletons.

# Towards counter-examples of injective types

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  Thus, the only type that is *provably* not injective is the empty type.
- But there are plenty of examples of types that cannot be shown to be injective in constructive mathematics, because their injectivity implies a **constructive taboo**: a statement that is not constructively provable and is false in some models.
- The relevant taboo in this case is weak excluded middle: for any proposition $P$, either $\neg P$ or $\neg\neg P$ holds.
  This is is equivalent to De Morgan's law.

# Counter-examples of injective types

- If any of the following types is injective, then weak excluded middle holds.
  - The type of booleans $\mathbf{2} :\equiv \mathbf{1} + \mathbf{1}$.
  - The simple types, obtained from $\mathbb{N}$ by iterating function types.
  - The type of Dedekind reals.
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  - More generally, any type with an apartness relation and two points apart.

    Recall: $\boxed{\text{apartness relation}} \approx \boxed{\text{positive (constructive) strengthening of } \neq}$.

- While the type $\Sigma(X : \mathcal{U}), X$ of **pointed** types and the type $\Sigma(X : \mathcal{U}), \neg\neg X$ of **non-empty** types are both injective, the type of **inhabited** types need not be.

  Prop. The type $\Sigma(X : \mathcal{U}), \|X\|$ of inhabited types is injective if and only if all propositions are *projective* (a weak choice principle).

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- Rice-like theorem: Injective types have no non-trivial decidable properties.

  Thm. If an injective type has a **decomposition**, then weak excluded middle holds.

  A *decomposition* of a type $X$ is defined to be a function $f : X \to \mathbf{2}$ such that we have $x_0 : X$ and $x_1 : X$ with $f\,x_0 = 0$ and $f\,x_1 = 1$.

# Future work

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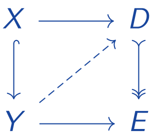

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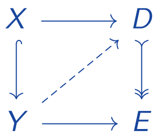


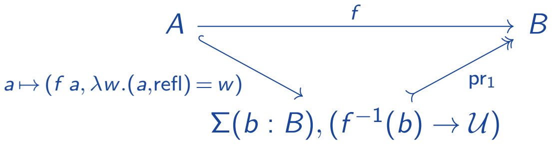

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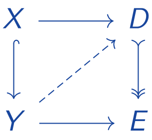


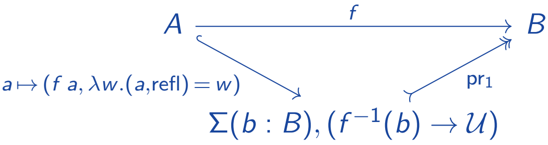


- Finish our preprint ☺

  For now, see the **TypeTopology/Agda** development: `https://www.cs.bham.ac.uk/~mhe/TypeTopology/InjectiveTypes.index.html`

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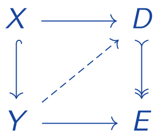


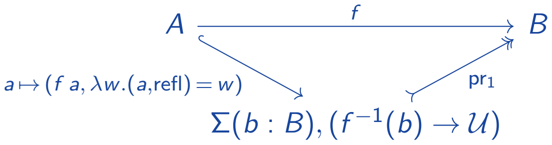


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Thank you!

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