Constructive Algebraic Completeness of First-Order Bi-intuitionistic Logic

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- Recent advances by Deakin/Shillito, Olkhovikov/Badia, Lyon/Shillito/Tiu

What is (propositional) bi-intuitionistic logic?

Syntax: $\varphi ::= p \in \mathbb{V} \mid \dot{\bot} \mid \varphi \mathbin{\dot{\land}} \varphi \mid \varphi \mathbin{\dot{\lor}} \varphi \mid \varphi \mathbin{\dot{\to}} \varphi \mid \varphi \mathbin{\dot{\prec}} \varphi$

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Axiomatic system: axioms for IL, (MP)

A_{10} $\quad \varphi \mathbin{\dot{\to}} (\psi \mathbin{\dot{\vee}} (\varphi \mathbin{\dot{\prec}} \psi))$

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Kripke semantics:

- Frames: preordered sets $(W, \leq)$ (i.e. $\leq$ is transitive and reflexive)
- Persistence: $\forall p \in \mathbb{V}. \forall w, v \in W.\ \ w \leq v \ \wedge\ w \in I(p) \ \to\ v \in I(p)$
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Conservative extension: $\mathrm{BIL}_{\dot{\bot}\,\dot{\wedge}\,\dot{\vee}\,\dot{\to}} = \mathrm{IL}$ (using LEM in the meta-theory)

Kripke completeness for BIL

Theorem

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- Analysis of constructive strength following Herbelin/K. 2023:

$$\text{Completeness}^{*} \quad \leftrightarrow \quad \forall X \subseteq \mathbb{N}.\, \neg\neg\forall n.\, (n \notin X) \vee \neg(n \notin X)$$

What is (first-order) bi-intuitionistic logic?

Syntax:

- Terms: $t ::= c \mid x \mid f(\vec{t})$
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Axiomatic system: axioms for BIL, (MP), (wDN), and

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$$A_{15} \quad \dot{\forall} x \varphi \mathbin{\dot{\to}} \varphi[t/x]$$

$$A_{16} \quad \varphi[t/x] \mathbin{\dot{\to}} \dot{\exists} x \varphi$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \dot{\forall} x \varphi} \text{ (Gen)} \qquad \frac{\Gamma \vdash \varphi \mathbin{\dot{\to}} \psi}{\Gamma \vdash \dot{\exists} x \varphi \mathbin{\dot{\to}} \psi} \text{ (EC)}$$

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Non-conservative extension: FOBIL proves *constant domain axiom*, FOIL does not

$$\dot{\forall} x(\varphi(x) \mathbin{\dot{\lor}} \psi) \mathbin{\dot{\to}} (\dot{\forall} x \varphi(x) \mathbin{\dot{\lor}} \psi)$$

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At least as soon as completeness is established...

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- Mechanisation in Rocq
- Using LEM, constructive status unclear

Algebraic semantics

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Definition

A (complete) bi-Heyting algebra is a (complete) Heyting algebra $(H, \leq, 0, \sqcap, \sqcup, \Rightarrow)$ with an additional binary operation $\prec\!\!\!=$ characterised by:

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Given a (complete) bi-Heyting algebra H, interpret formulas of (FO)BIL:

$$[\![\varphi \dot{\prec} \psi]\!] := [\![\varphi]\!] \prec [\![\psi]\!] \qquad [\![P\vec{t}]\!] := I(P\vec{t}) \qquad [\![\dot{\forall} x \varphi]\!] := \prod_{t:\mathsf{tm}} [\![\varphi[t/x]]\!]$$

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Fact

(FO)BIL *is sound for (complete) bi-Heyting algebras.*

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Proof.

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BIL *is complete for bi-Heyting algebras.*

Proof.

- Consider the Lindenbaum algebra $L = (\mathsf{fm}, \cdot \vdash \cdot, \dot{\bot}, \dot{\wedge}, \dot{\vee}, \dot{\rightarrow}, \dot{\prec})$.

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BIL *is complete for bi-Heyting algebras.*

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- Consider the Lindenbaum algebra $L = (\mathrm{fm}, \cdot \vdash \cdot, \dot{\bot}, \dot{\wedge}, \dot{\vee}, \dot{\rightarrow}, \dot{\prec})$.
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- Conclude completeness. □

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- Strong completeness for algebraic semantics
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Thank you!

Bibliography