In Cantor Space, no one can hear you Stream

There is no alien way to compute

Martin Baillon

June 09, 2025

Continuity of all MLTT-definable Functionals

Continuity of all MLTT-definable Functionals

Continuity of all MLTT-definable Functionals

Continuity of all MLTT-definable Functionals

Continuity of all MLTT-definable Functionals

MLTT-definable Functionals

MLTT-definable Functionals

MLTT-definable Functionals

Π

MLTT-definable Functionals

Π Σ

MLTT-definable Functionals

$\Pi \quad \Sigma \quad \mathbb{B}$

MLTT-definable Functionals

$\Pi \quad \Sigma \quad \mathbb{B} \quad \mathbb{N}$

MLTT-definable Functionals

$\Pi \quad \Sigma \quad \mathbb{B} \quad \mathbb{N} \quad =$

MLTT-definable Functionals

$\Pi \quad \Sigma \quad \mathbb{B} \quad \mathbb{N} \quad = \quad \square_0$

MLTT-definable Functionals

MLTT-definable Functionals

$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

MLTT is normalizing so F can only ask finitely many queries.

$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

MLTT is normalizing so F can only ask finitely many queries.

$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

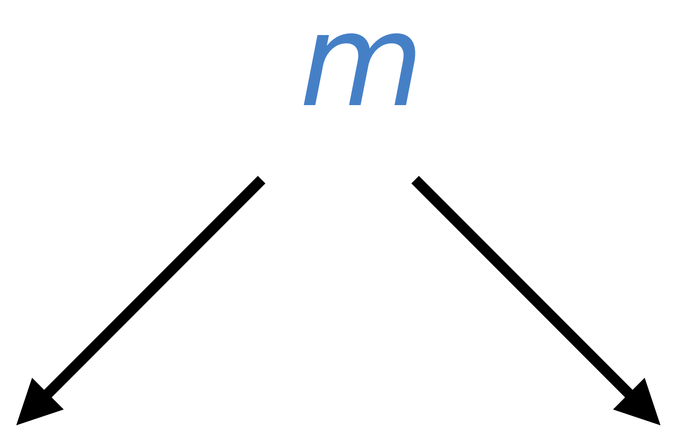
MLTT is normalizing so F can only ask finitely many queries.

m

$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

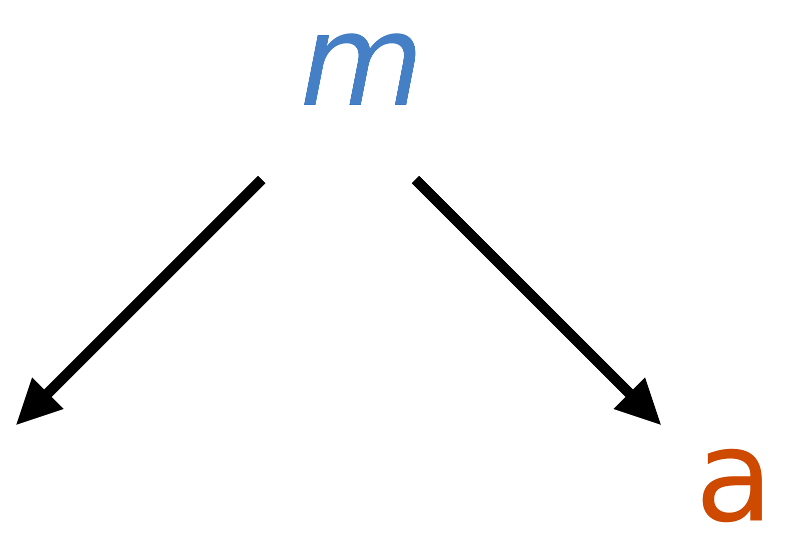
MLTT is normalizing so F can only ask finitely many queries.



$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

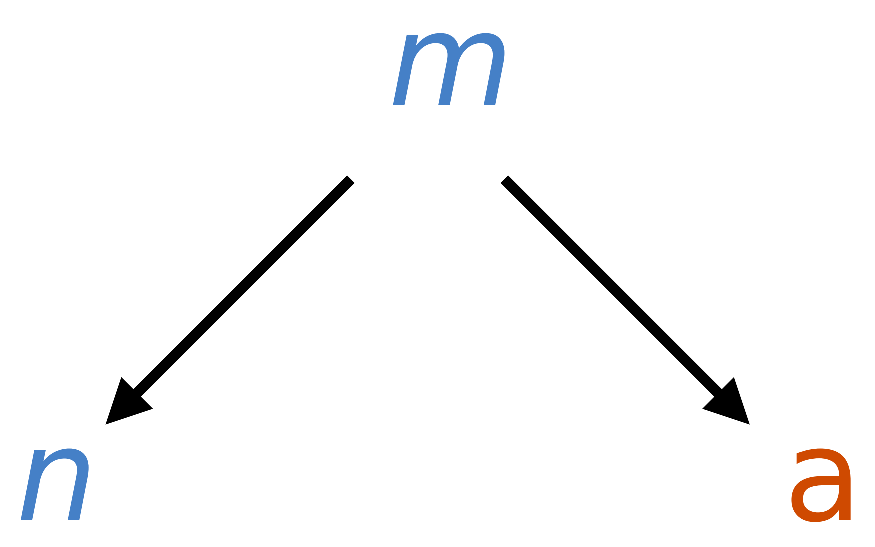
MLTT is normalizing so F can only ask finitely many queries.



$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

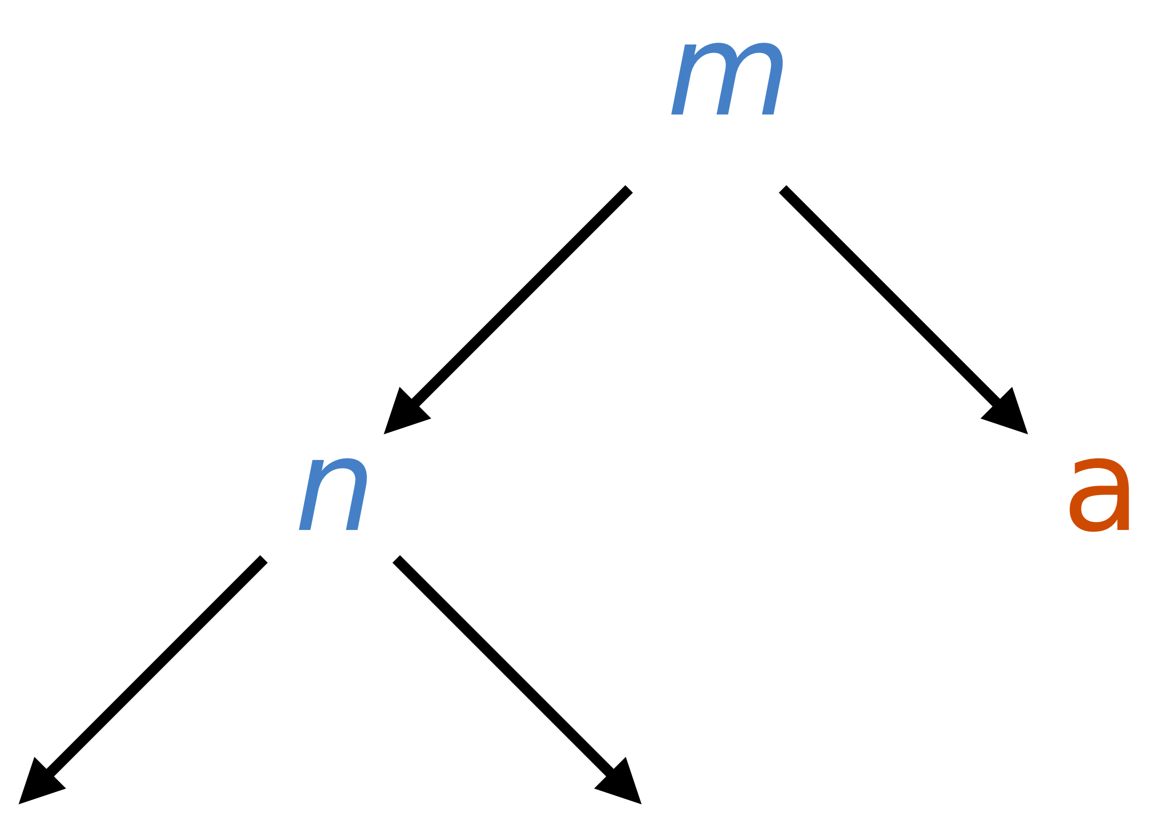
MLTT is normalizing so F can only ask finitely many queries.



$$\cdot \vdash_{\mathrm{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

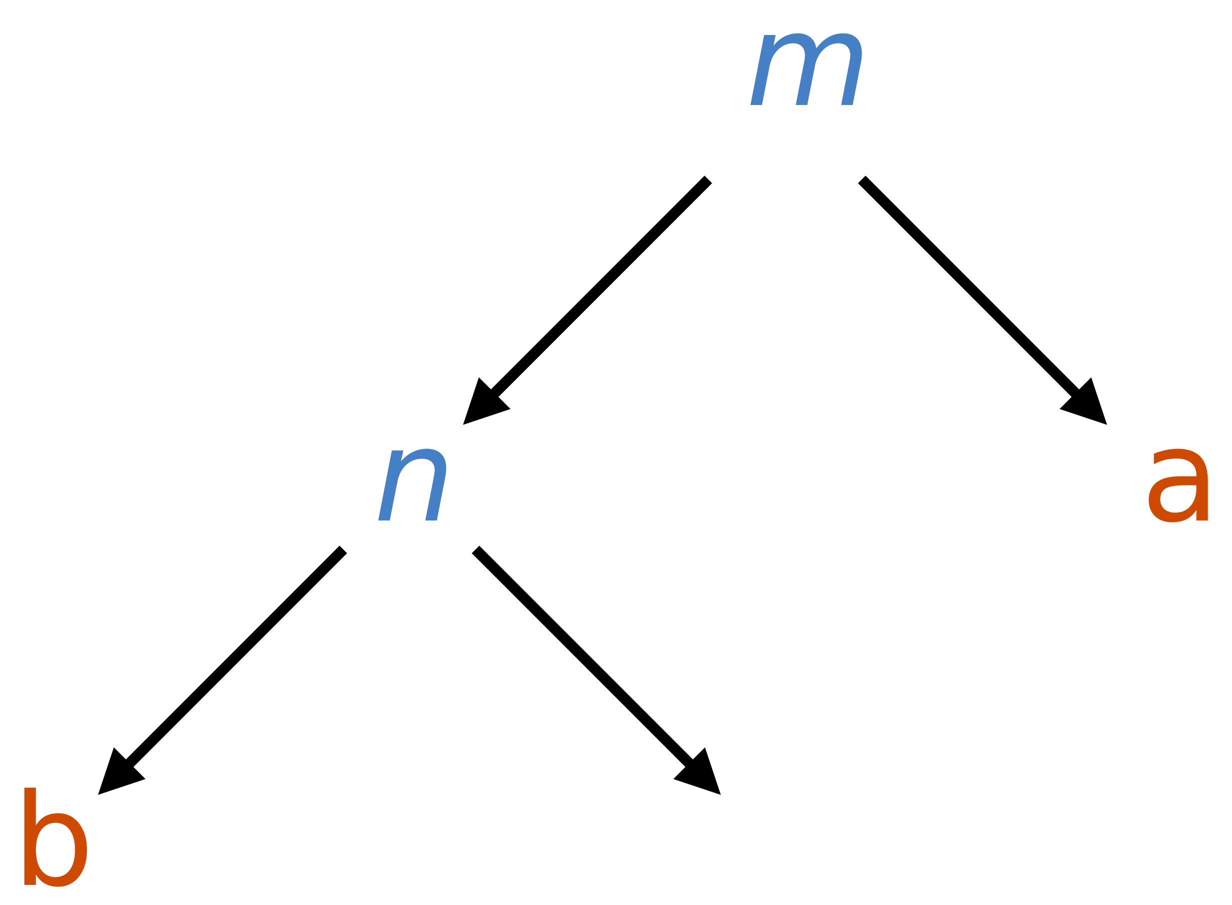
MLTT is normalizing so F can only ask finitely many queries.



$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

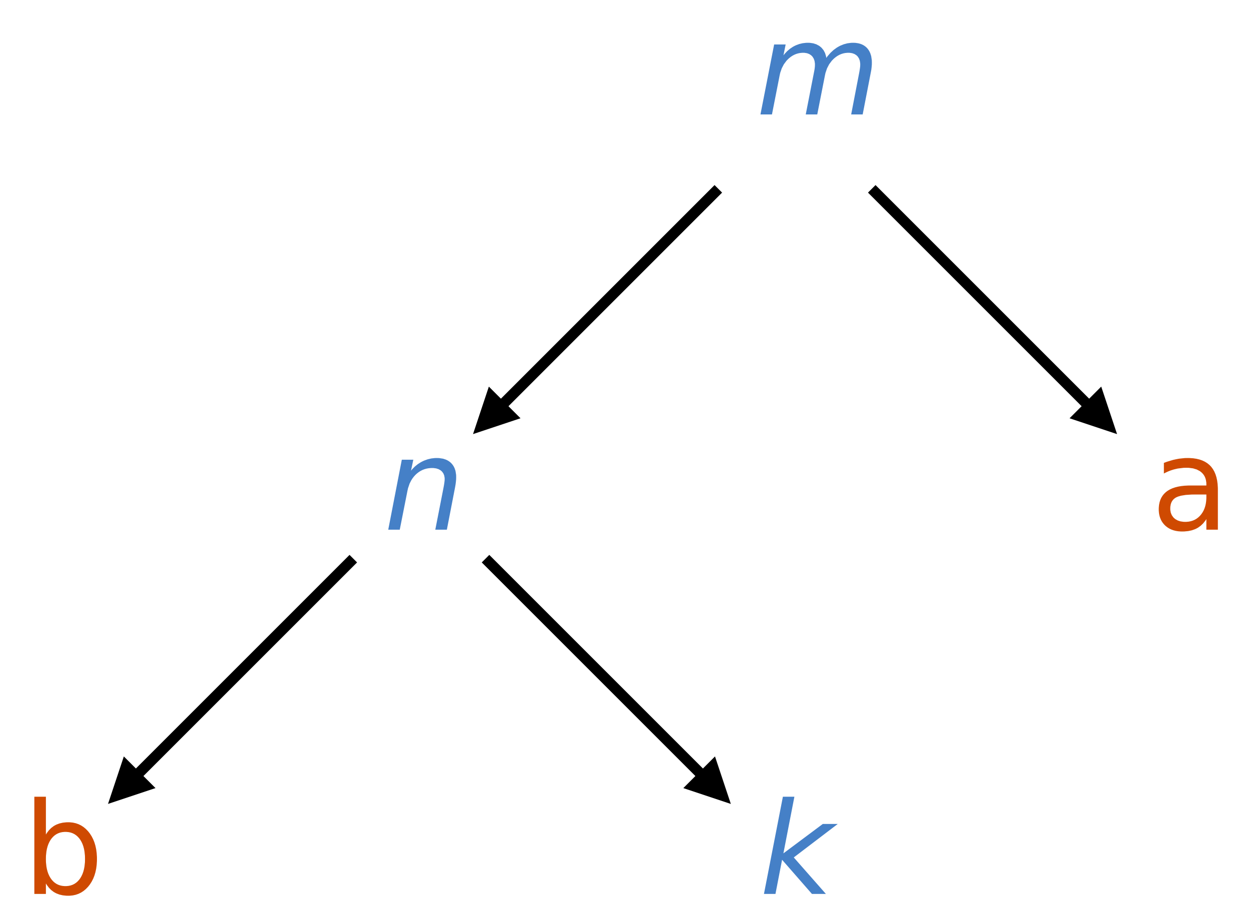
MLTT is normalizing so F can only ask finitely many queries.



$$\cdot \vdash_{\text{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

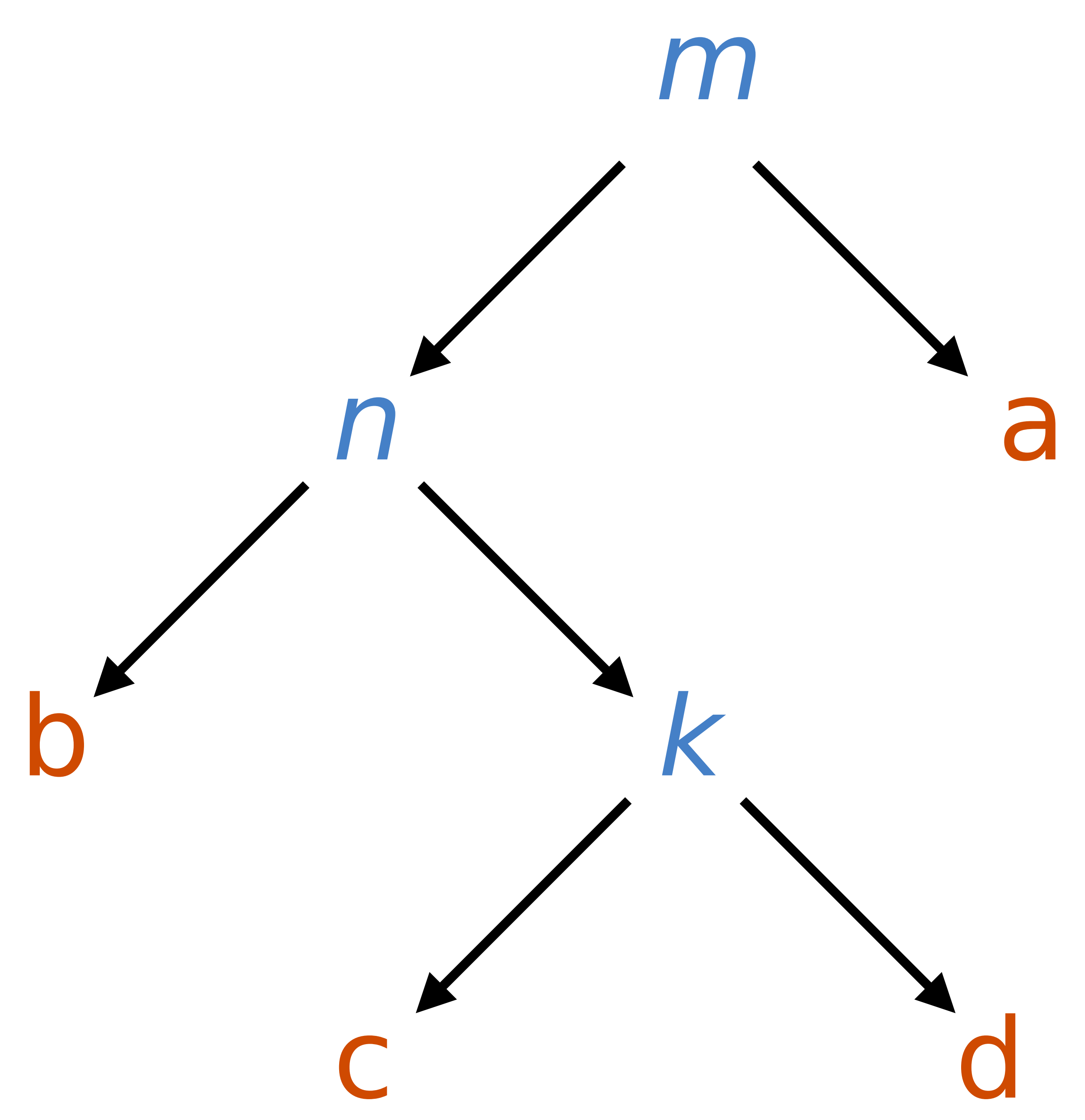
MLTT is normalizing so F can only ask finitely many queries.



$$\cdot \vdash_{\mathsf{MLTT}} F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

In MLTT, any $\alpha : \mathbb{N} \to \mathbb{B}$ is a black-box for F.

MLTT is normalizing so F can only ask finitely many queries.



Continuity

Continuity

Talking trees

We consider the following **Dialogue** operator :

$$\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad := \\ \mid \quad \eta : A \to \mathfrak{D}\ A \\ \mid \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$$

Continuity

Talking trees

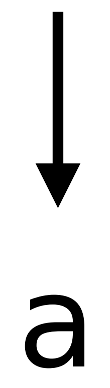
We consider the following **Dialogue** operator :

$$\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad := $$
$$\mid \quad \eta : A \to \mathfrak{D}\ A$$
$$\mid \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$$

$\mathfrak{D}\ A$ is the type of well-founded, $\mathbb{B}$-branching trees, with inner nodes labeled in $\mathbb{N}$ and leaves in A.

Continuity

Talking trees



We consider the following **Dialogue** operator :

$$\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad :=$$
$$\mid \quad \eta : A \to \mathfrak{D}\ A$$
$$\mid \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$$

$\mathfrak{D}\ A$ is the type of well-founded, $\mathbb{B}$-branching trees, with inner nodes labeled in $\mathbb{N}$ and leaves in A.

Continuity

Talking trees

We consider the following **Dialogue** operator :

$$
\begin{array}{l}
\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad := \\
\quad | \quad \eta : A \to \mathfrak{D}\ A \\
\quad | \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.
\end{array}
$$

$\mathfrak{D}\ A$ is the type of well-founded, $\mathbb{B}$-branching trees, with inner nodes labeled in $\mathbb{N}$ and leaves in A.

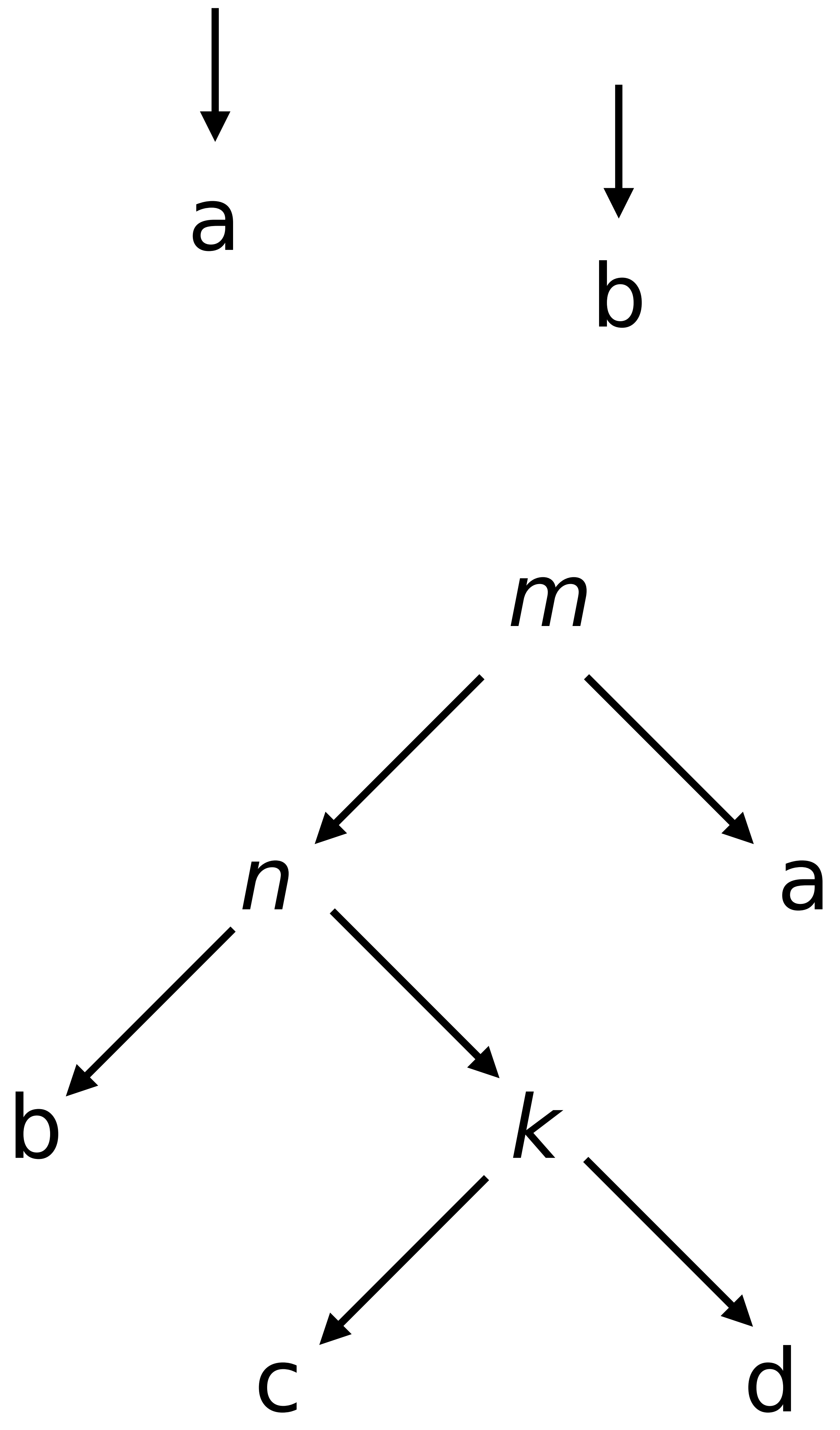
Continuity

Talking trees

We consider the following **Dialogue** operator :

$$\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad :=$$
$$\mid \quad \eta : A \to \mathfrak{D}\ A$$
$$\mid \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$$

$\mathfrak{D}\ A$ is the type of well-founded, $\mathbb{B}$-branching trees, with inner nodes labeled in $\mathbb{N}$ and leaves in A.



Inductive $\mathfrak{D}\ (A : \square) : \square \quad :=$

$\mid \quad \eta : A \to \mathfrak{D}\ A$

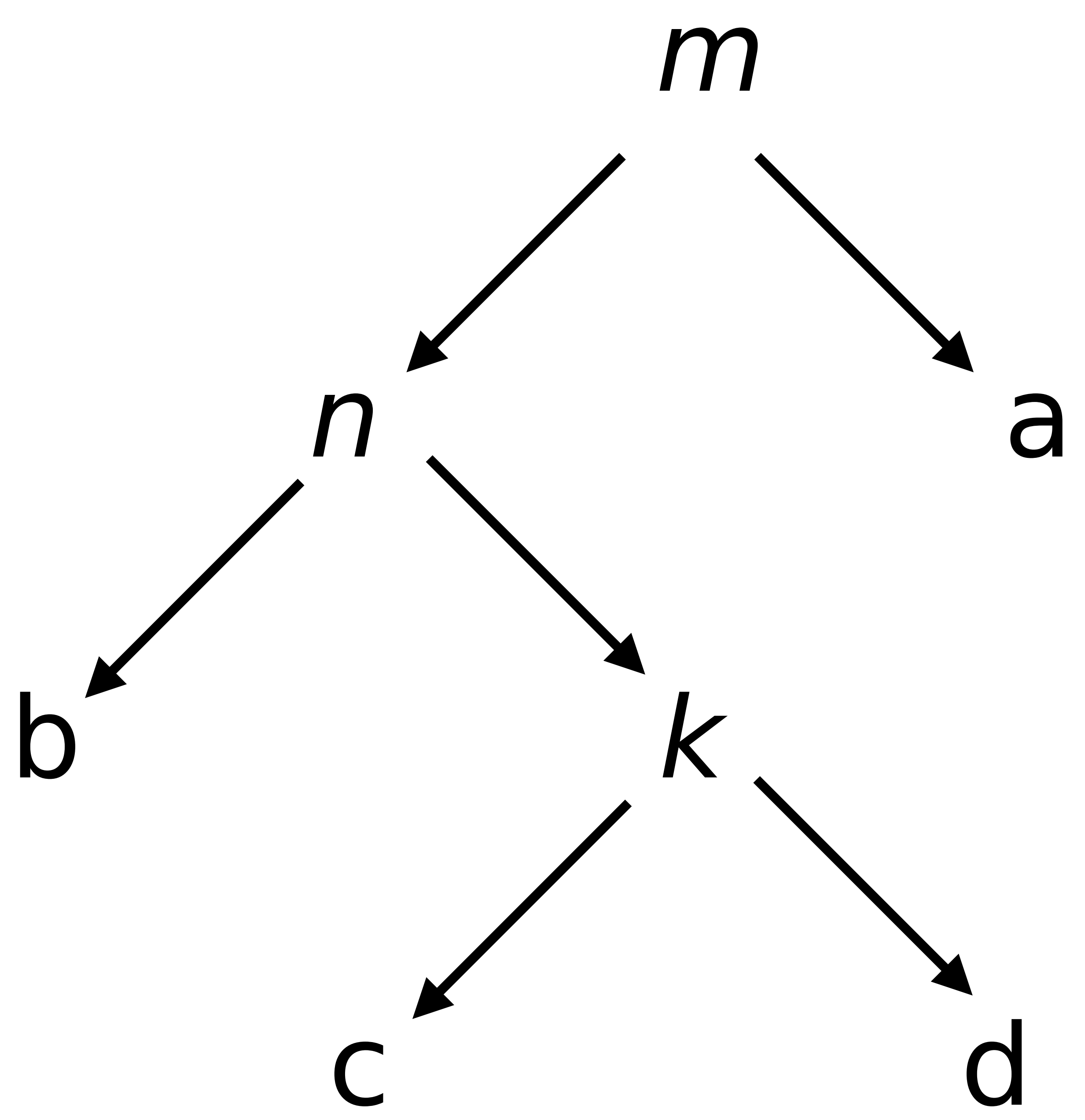
$\mid \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$

Continuity

Every tree will die a log

We consider the following **decode** function :

$$
\begin{array}{lcl}
\partial & : & \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\
\partial\ (\eta\ x)\ \alpha & := & x \\
\partial\ (\beta\ i\ k)\ \alpha & := & \partial\ (k\ (\alpha\ i))\ \alpha
\end{array}
$$



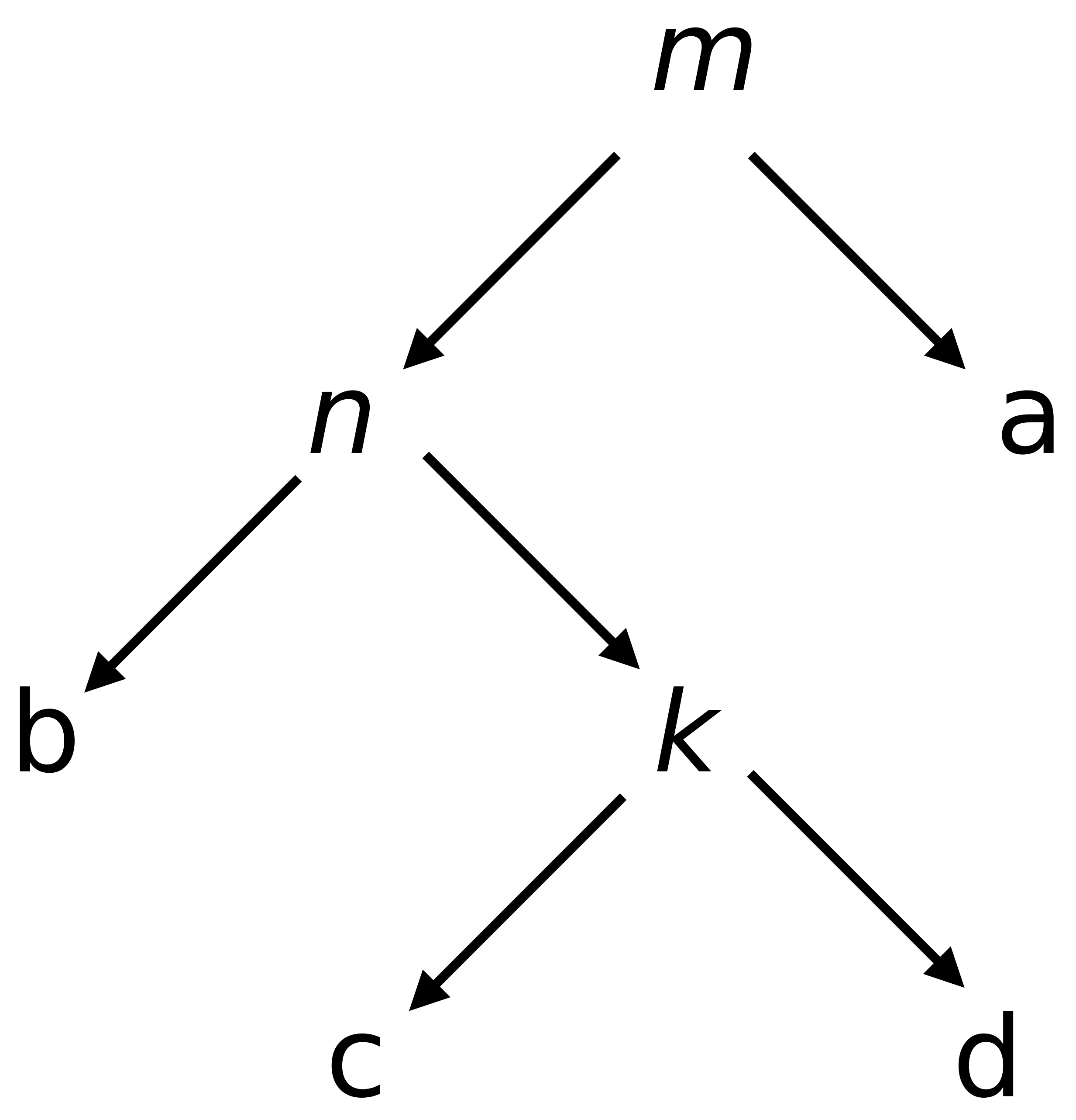
```
Inductive 𝔇 (A : □) : □  :=
  | η : A → 𝔇 A
  | β : ℕ → (𝔹 → 𝔇 A) → 𝔇 A.
```

Continuity

Every tree will die a log

We consider the following **decode** function :

$$
\begin{array}{lcl}
\partial & : & \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\
\partial\ (\eta\ x)\ \alpha & := & x \\
\partial\ (\beta\ i\ k)\ \alpha & := & \partial\ (k\ (\alpha\ i))\ \alpha
\end{array}
$$



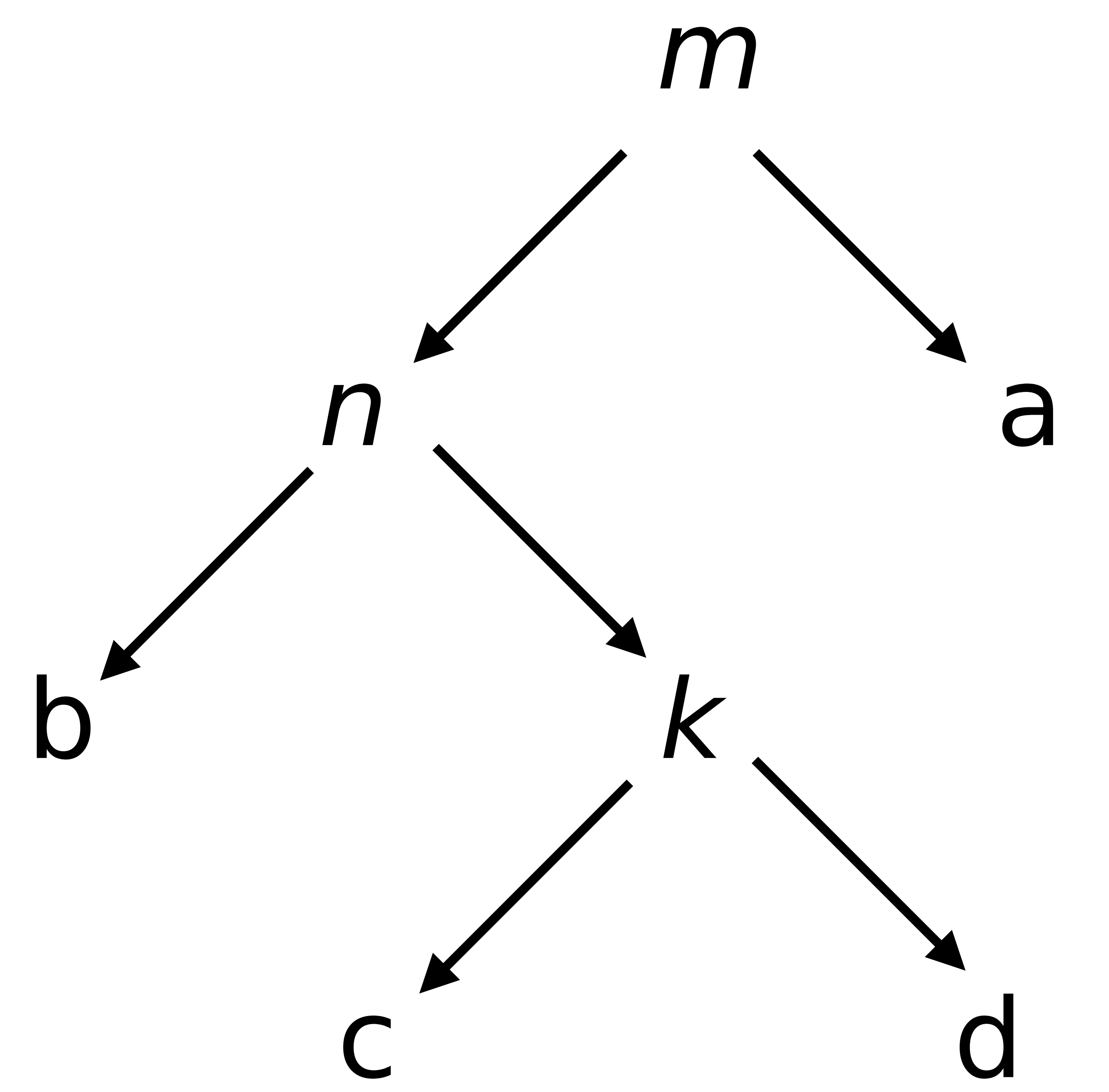
Continuity

Every tree will die a log

Inductive $\mathfrak{D}$ $(A : \square) : \square$:=
| $\eta : A \to \mathfrak{D}\ A$
| $\beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$

We consider the following **decode** function :

$$\begin{array}{lcl} \partial & : & \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\ \partial\ (\eta\ x)\ \alpha & := & x \\ \partial\ (\beta\ i\ k)\ \alpha & := & \partial\ (k\ (\alpha\ i))\ \alpha \end{array}$$



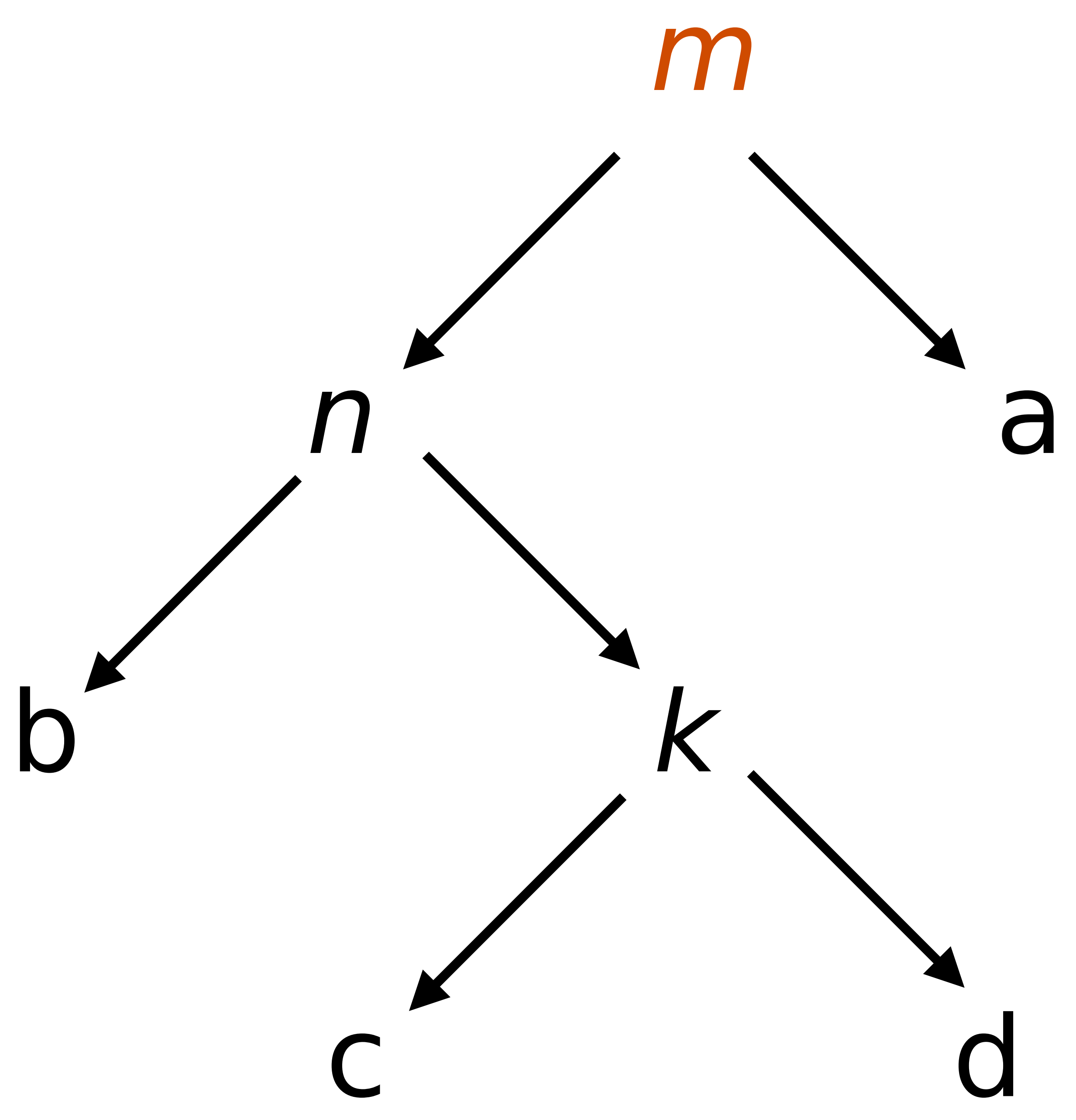
$$\begin{array}{l} \texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad := \\ \quad |\quad \eta : A \to \mathfrak{D}\ A \\ \quad |\quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A. \end{array}$$

Continuity

Every tree will die a log

We consider the following **decode** function :

$$\begin{array}{lcl} \partial & : & \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\ \partial\ (\eta\ x)\ \alpha & := & x \\ \partial\ (\beta\ i\ k)\ \alpha & := & \partial\ (k\ (\alpha\ i))\ \alpha \end{array}$$



$$
\begin{aligned}
&\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad := \\
&\quad \mid \ \eta : A \to \mathfrak{D}\ A \\
&\quad \mid \ \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.
\end{aligned}
$$

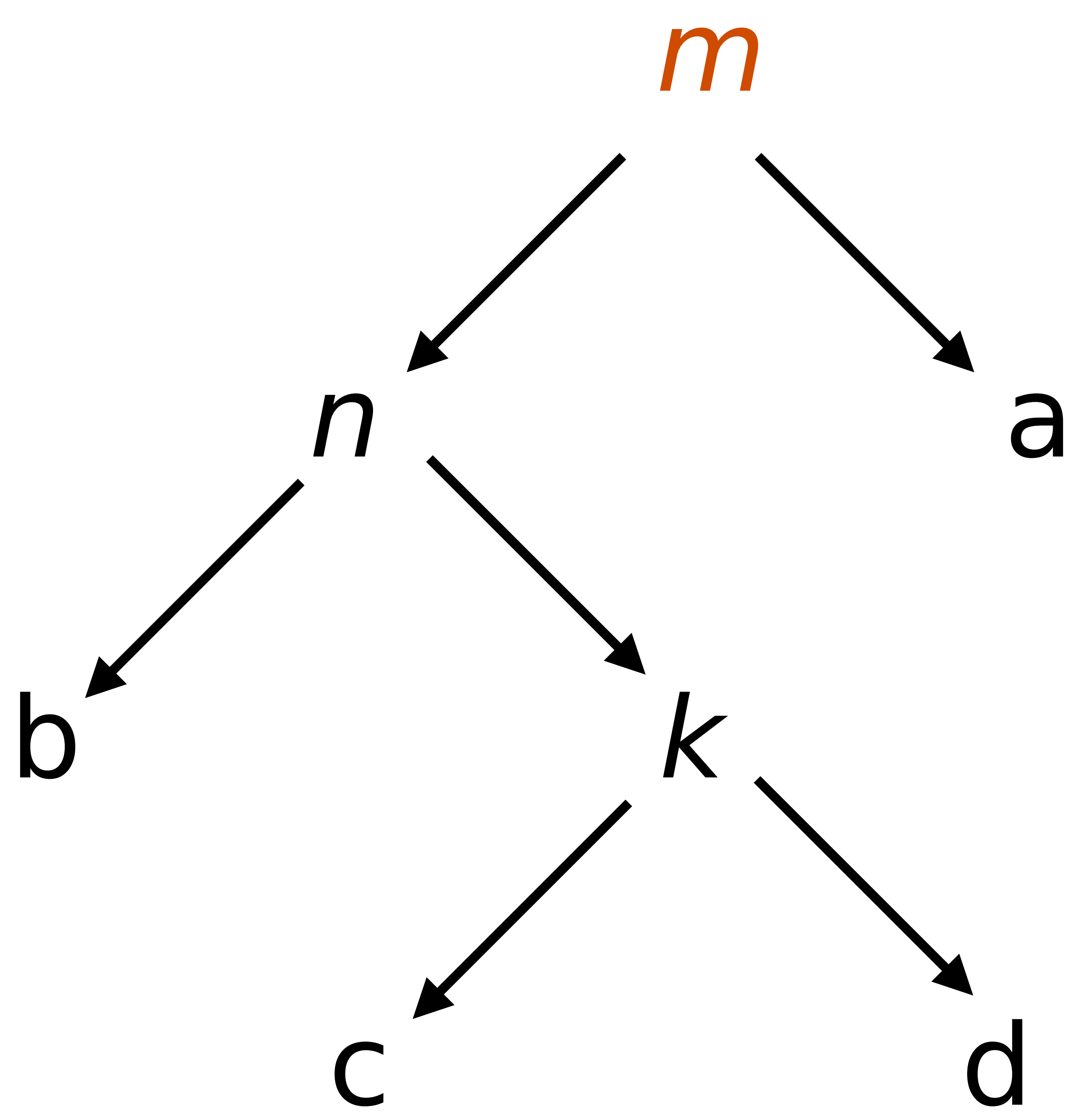
Continuity

Every tree will die a log

We consider the following **decode** function :

$$
\begin{aligned}
&\partial &&: &&\Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\
&\partial\ (\eta\ x)\ \alpha &&:= &&x \\
&\partial\ (\beta\ i\ k)\ \alpha &&:= &&\partial\ (k\ (\alpha\ i))\ \alpha
\end{aligned}
$$

$$\alpha\ m \ = \ \texttt{true}$$



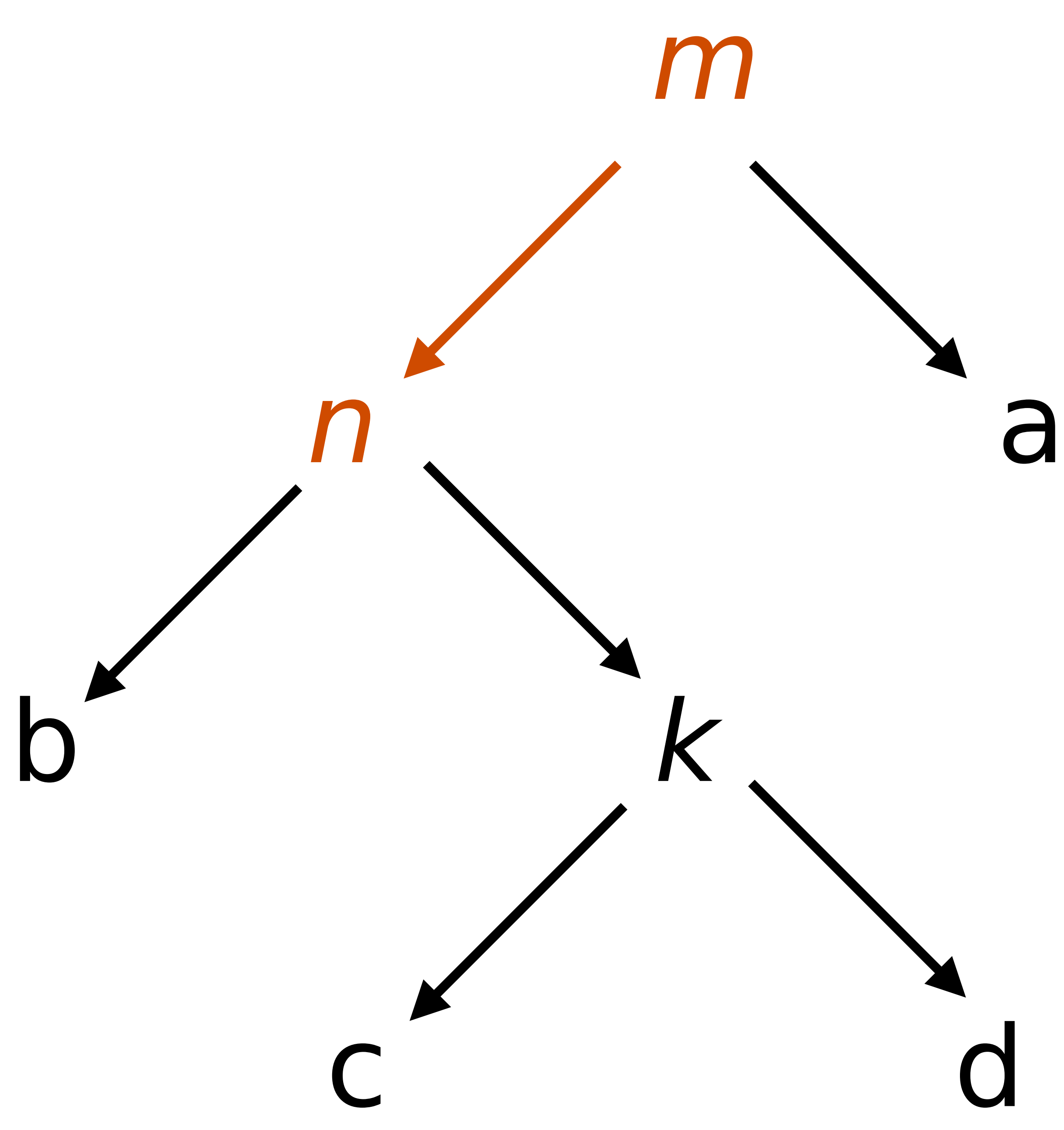
$$\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad := $$
$$\mid \quad \eta : A \to \mathfrak{D}\ A$$
$$\mid \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$$

Continuity

Every tree will die a log

We consider the following **decode** function :

$$\begin{array}{lcl} \partial & : & \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\ \partial\ (\eta\ x)\ \alpha & := & x \\ \partial\ (\beta\ i\ k)\ \alpha & := & \partial\ (k\ (\alpha\ i))\ \alpha \end{array}$$



$$\alpha\ m \ = \ \texttt{true}$$

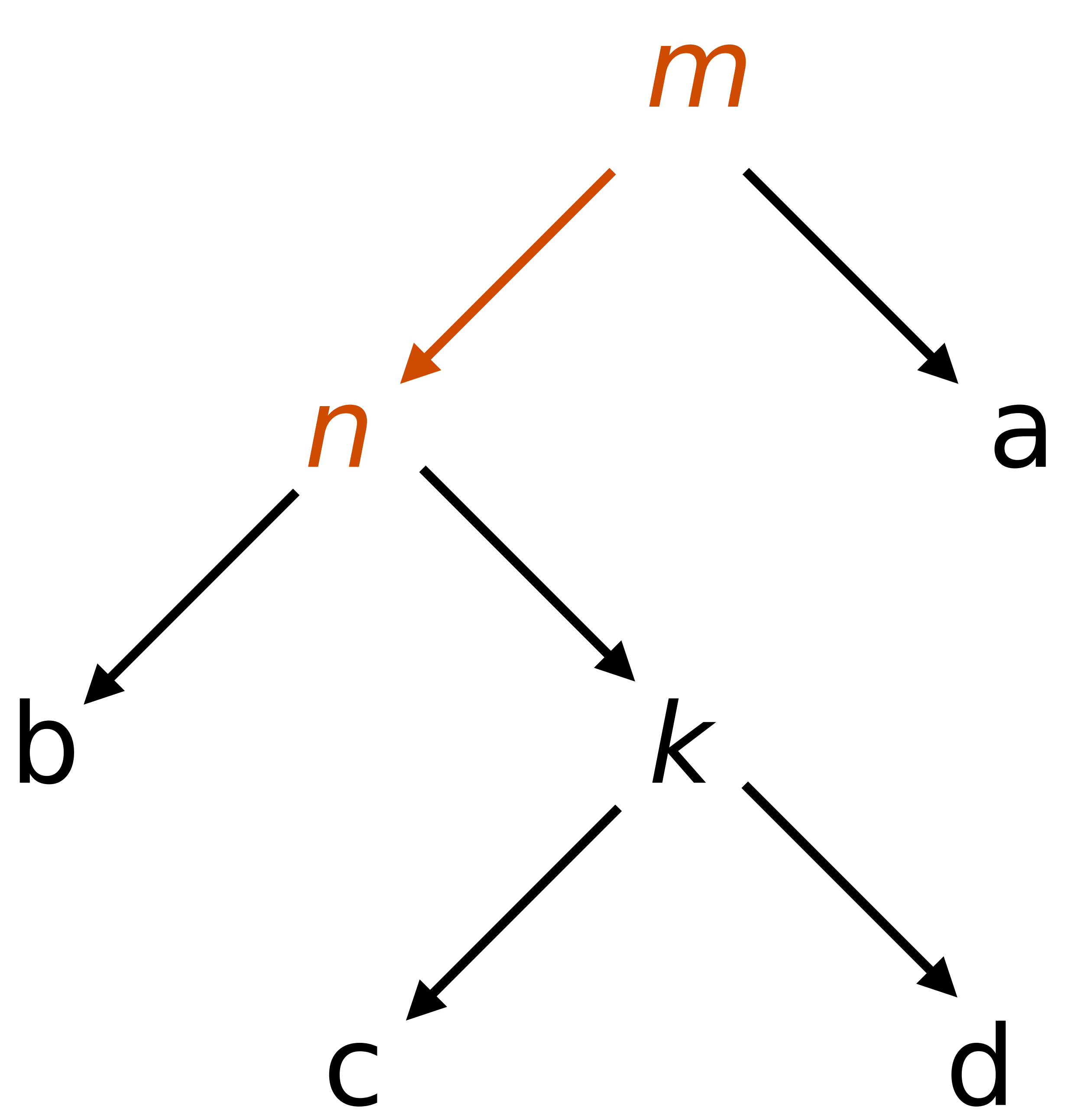
Continuity

$$\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad :=$$
$$\mid\ \eta : A \to \mathfrak{D}\ A$$
$$\mid\ \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$$

Every tree will die a log

We consider the following **decode** function :

$$\begin{aligned}
\partial \quad & : \quad \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\
\partial\ (\eta\ x)\ \alpha \quad & := \quad x \\
\partial\ (\beta\ i\ k)\ \alpha \quad & := \quad \partial\ (k\ (\alpha\ i))\ \alpha
\end{aligned}$$



$$\alpha\ m \ = \ \texttt{true}$$
$$\alpha\ n \ = \ \texttt{false}$$

Inductive $\mathfrak{D}\ (A : \square) : \square\ :=$
$\mid\ \eta : A \to \mathfrak{D}\ A$
$\mid\ \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$

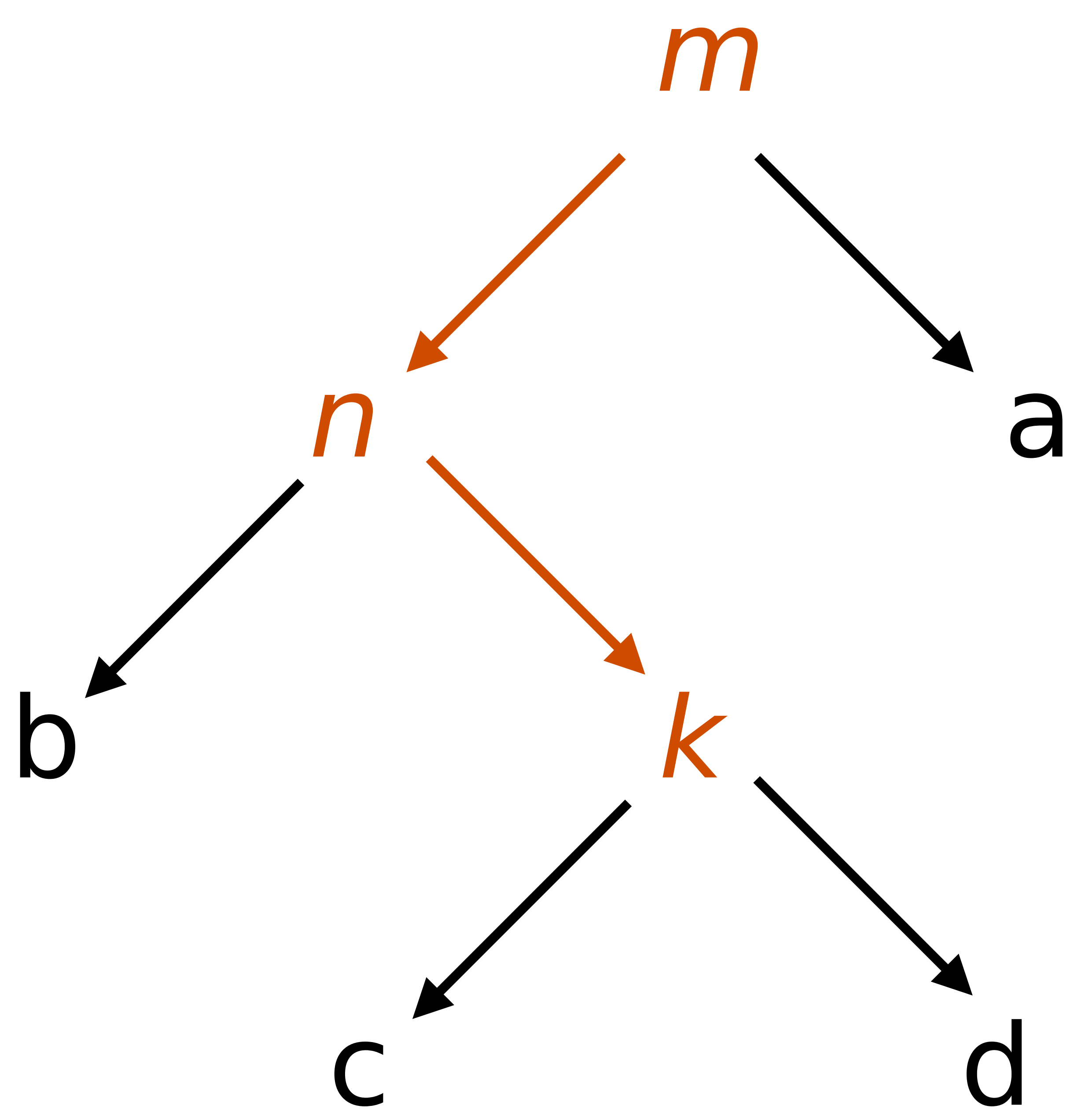
Continuity

Every tree will die a log

We consider the following **decode** function :

$$
\begin{array}{lll}
\partial & : & \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\
\partial\ (\eta\ x)\ \alpha & := & x \\
\partial\ (\beta\ i\ k)\ \alpha & := & \partial\ (k\ (\alpha\ i))\ \alpha
\end{array}
$$

$$
\begin{array}{lll}
\alpha\ m & = & \texttt{true} \\
\alpha\ n & = & \texttt{false}
\end{array}
$$



Continuity

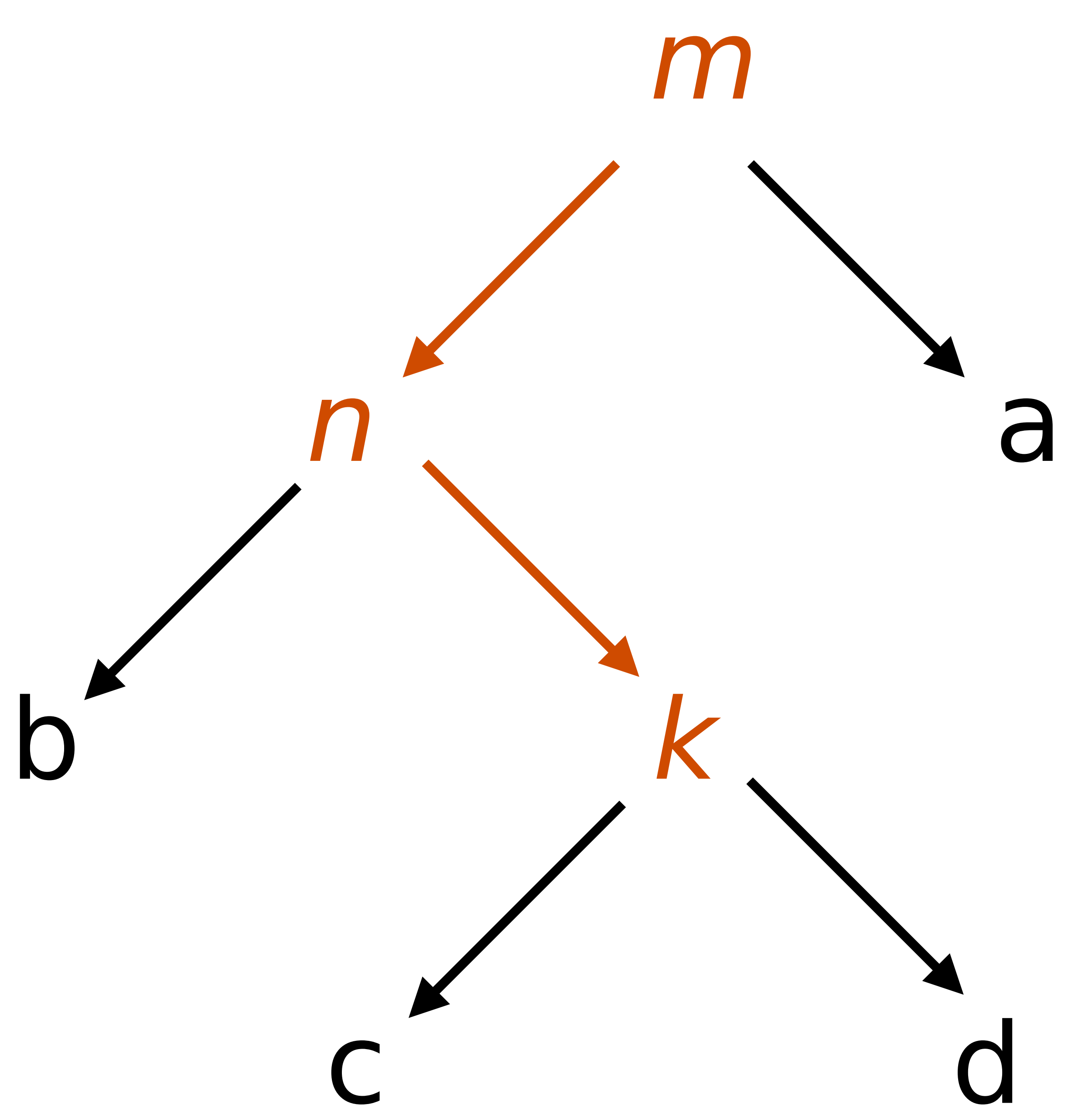
Every tree will die a log

$$\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad :=$$
$$\mid \quad \eta : A \to \mathfrak{D}\ A$$
$$\mid \quad \beta : \mathbb{N} \to (\mathbb{B} \to \mathfrak{D}\ A) \to \mathfrak{D}\ A.$$

We consider the following **decode** function :

$$\begin{array}{lcl}
\partial & : & \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\ A \\
\partial\ (\eta\ x)\ \alpha & := & x \\
\partial\ (\beta\ i\ k)\ \alpha & := & \partial\ (k\ (\alpha\ i))\ \alpha
\end{array}$$

$$\begin{array}{lcl}
\alpha\ m & = & \texttt{true} \\
\alpha\ n & = & \texttt{false} \\
\alpha\ k & = & \texttt{true}
\end{array}$$



$$
\begin{aligned}
&\texttt{Inductive}\ \mathfrak{D}\ (A : \square) : \square \quad := \\
&\quad |\quad \eta : A \rightarrow \mathfrak{D}\ A \\
&\quad |\quad \beta : \mathbb{N} \rightarrow (\mathbb{B} \rightarrow \mathfrak{D}\ A) \rightarrow \mathfrak{D}\ A.
\end{aligned}
$$

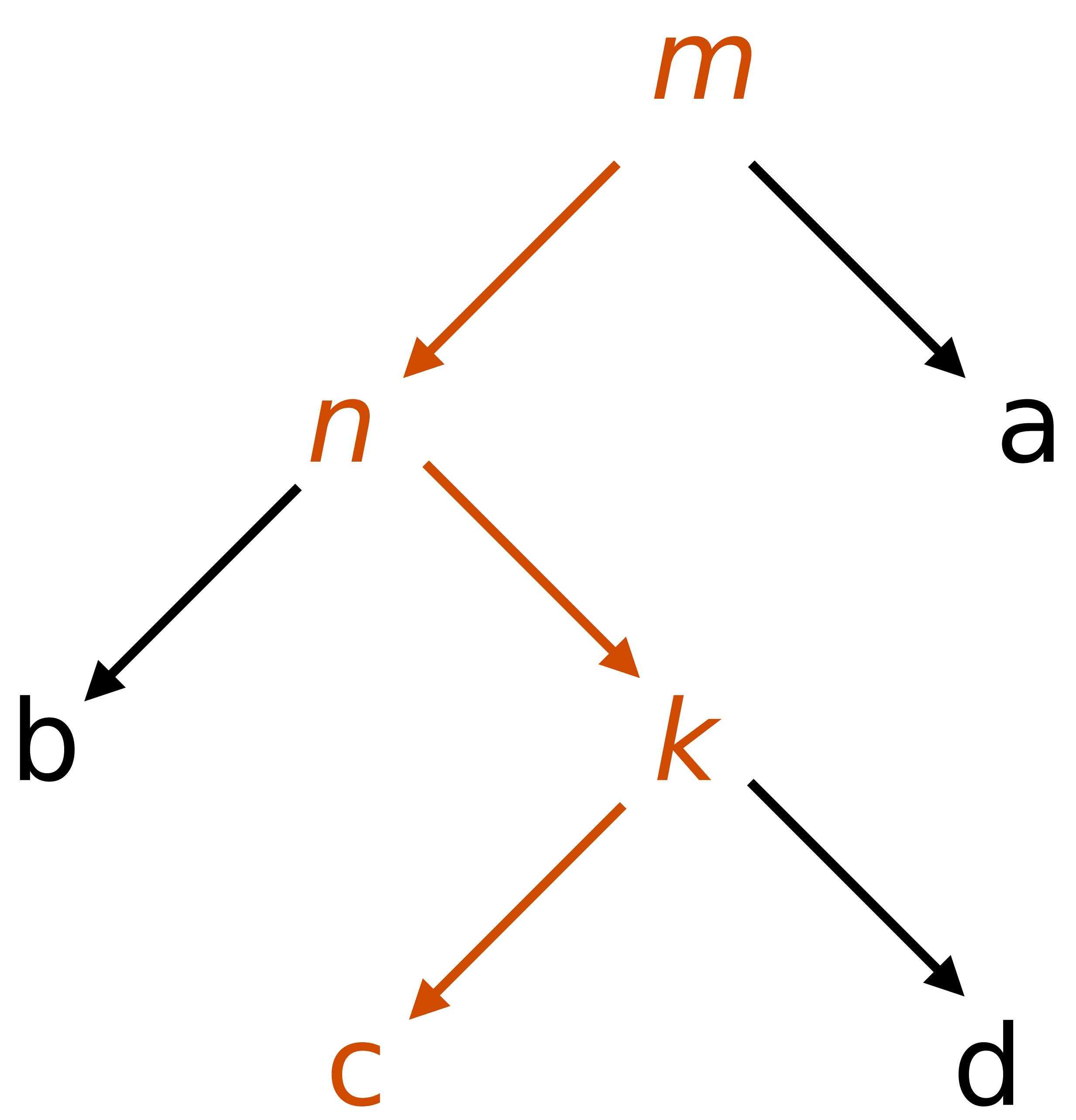
Continuity

Every tree will die a log

We consider the following **decode** function :

$$
\begin{aligned}
&\partial && : && \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \rightarrow \mathbb{B}).\ A \\
&\partial\ (\eta\ x)\ \alpha && := && x \\
&\partial\ (\beta\ i\ k)\ \alpha && := && \partial\ (k\ (\alpha\ i))\ \alpha
\end{aligned}
$$

$$
\begin{aligned}
\alpha\ m &= \texttt{true} \\
\alpha\ n &= \texttt{false} \\
\alpha\ k &= \texttt{true}
\end{aligned}
$$



Continuity

```
Inductive 𝔇 (A : □) : □  :=
  | η : A → 𝔇 A
  | β : ℕ → (𝔹 → 𝔇 A) → 𝔇 A.
```

Every tree will die a log

We consider the following **decode** function :

$$\begin{aligned}
\partial \quad &: \quad \Pi\{A : \square\}\,(d : \mathfrak{D}\ A)\,(\alpha : \mathbb{N} \to \mathbb{B}).\, A \\
\partial\ (\eta\ x)\ \alpha \quad &:= \quad x \\
\partial\ (\beta\ i\ k)\ \alpha \quad &:= \quad \partial\ (k\ (\alpha\ i))\ \alpha
\end{aligned}$$

Definition

A function $f : (\mathbb{N} \to \mathbb{B}) \to A$ is said **dialogue continuous** if:

$$\Sigma d : \mathfrak{D}\, A.\, \Pi\alpha : (\mathbb{N} \to \mathbb{B}).\, f\ \alpha = \partial\ d\ \alpha$$

Continuity of MLTT-definable Functionals

Continuity of all MLTT-definable Functionals

Continuity of all MLTT-definable Functionals

Continuity principle

$$\frac{}{\cdot \vdash \mathcal{H} : \Pi(F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}).\, \mathcal{C}_{\mathfrak{D}}\ F}$$

Continuity of all MLTT-definable Functionals

Continuity principle

$$\frac{}{\cdot \vdash \mathcal{H} : \Pi(F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}).\, \mathcal{C}_{\mathfrak{D}}\ F}$$

Continuity rule

$$\frac{\cdot \vdash F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}}{\cdot \vdash \mathcal{H}_F : \mathcal{C}_{\mathfrak{D}}\ F}$$

Continuity of all MLTT-definable Functionals

Continuity principle

$$\frac{}{\cdot \vdash \mathcal{H} : \Pi(F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}).\, \mathcal{C}_{\mathfrak{D}}\ F}$$

Continuity rule

$$\frac{\cdot \vdash F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}}{\cdot \vdash \mathcal{H}_F : \mathcal{C}_{\mathfrak{D}}\ F}$$

External continuity

If

$$\cdot \vdash F : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

then we get a meta dialogue tree $d : \mathfrak{D}\ \mathbb{N}$ such that if

$$\cdot \vdash M : \mathbb{N} \to \mathbb{B}, \qquad \text{then} \qquad \cdot \vdash F\ M : \mathbb{N}$$

reduces to what we compute with d and M.

Battle plan

Battle plan

Define a well-chosen type theory

Battle plan

Define a well-chosen type theory

Prove normalization of this theory

Battle plan

Define a well-chosen type theory

Prove normalization of this theory

Derive continuity from a modified canonicity theorem

Split Type Theory

Split Type Theory

Brute forcing

Split Type Theory

Brute forcing

$$\digamma TT$$

Split Type Theory

Brute forcing

$$\mathfrak{s}\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$$

Split Type Theory

$$\mathfrak{f}\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$$

Brute forcing

$$\Gamma \mid \ell \vdash_{\mathfrak{f}\mathrm{TT}} t : A$$

Split Type Theory

$$\mathfrak{s}\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$$

Brute forcing

$$\Gamma \mid \ell \vdash_{\mathfrak{s}\mathrm{TT}} t : A$$

Split Type Theory

$\digamma\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$

Brute forcing

$$\Gamma \mid \ell \vdash_{\digamma\mathrm{TT}} t : A$$

$\ell : \mathrm{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

$$\mathfrak{s}\mathsf{TT} := \mathsf{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$$

Brute forcing

$$\Gamma \mid \ell \vdash_{\mathfrak{s}\mathsf{TT}} \mathfrak{f}\ \overline{\mathsf{n}}$$

$\ell : \mathsf{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

$$\digamma\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$$

Brute forcing

$$\Gamma \mid \ell \vdash_{\digamma\mathrm{TT}} \mathfrak{f}\ \overline{\mathtt{n}} \equiv \mathtt{true} : \mathbb{B} \text{ if } (\mathrm{n}, \mathtt{true}) \in \ell$$

$\ell : \mathrm{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

$$\mathfrak{f}\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$$

Brute forcing

$$\Gamma \mid \ell \vdash_{\mathfrak{f}\mathrm{TT}} \mathfrak{f}\ \overline{\mathtt{n}} \equiv \mathtt{false} : \mathbb{B} \text{ if } (\mathrm{n}, \mathtt{false}) \in \ell$$

$\ell : \mathrm{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

$\digamma TT := MLTT + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$

Brute forcing

$$\Gamma \mid \ell \vdash_{\digamma TT} \mathfrak{f}\ \overline{\mathtt{n}} \equiv \mathtt{false} : \mathbb{B} \text{ if } (\mathtt{n}, \mathtt{false}) \in \ell$$

$\ell : \mathrm{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

$\mathfrak{s}\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$

Brute forcing

meta $\mathbb{N}$

$$\Gamma \mid \ell \vdash_{\mathfrak{s}\mathrm{TT}} \mathfrak{f}\ \overline{\mathtt{n}} \equiv \mathtt{false} : \mathbb{B} \text{ if } (\mathtt{n}, \mathtt{false}) \in \ell$$

$\ell : \mathrm{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

$$\mathfrak{s}\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$$

Brute forcing

meta $\mathbb{N}$

$$\Gamma \mid \ell \vdash_{\mathfrak{s}\mathrm{TT}} \mathfrak{f}\ \overline{\mathtt{n}} \equiv \mathtt{false} : \mathbb{B} \text{ if } (\mathtt{n}, \mathtt{false}) \in \ell$$

$\ell : \mathrm{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

$\mathfrak{f}\mathrm{TT} := \mathrm{MLTT} + \mathfrak{f} : \mathbb{N} \to \mathbb{B}$

Brute forcing

term of $\mathfrak{f}\mathrm{TT}$ meta $\mathbb{N}$

$$\Gamma \mid \ell \vdash_{\mathfrak{f}\mathrm{TT}} \mathfrak{f}\ \overline{\mathtt{n}} \equiv \mathtt{false} : \mathbb{B} \text{ if } (\mathtt{n}, \mathtt{false}) \in \ell$$

$\ell : \mathrm{list}\ (\mathbb{N} \times \mathbb{B})$ is the current state of knowledge about $\mathfrak{f}$

Split Type Theory

Amazing new equations

$$\frac{}{\Gamma \mid \ell \vdash_{\mathfrak{F}\mathsf{TT}} t \equiv u : A}$$

Split Type Theory

Amazing new equations

$$\frac{}{\Gamma \mid \ell \vdash_{\mathsf{F}\mathrm{TT}} t \equiv u : A}\ \mathrm{n} \notin \mathrm{dom}(\ell)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (\mathtt{n}, \mathtt{true}) :: \ell \vdash_{\mathsf{F}\mathsf{TT}} t \equiv u : A}{\Gamma \mid \ell \vdash_{\mathsf{F}\mathsf{TT}} t \equiv u : A}\ \mathtt{n} \notin \mathrm{dom}(\ell)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (\mathtt{n}, \mathtt{true}) :: \ell \vdash_{\digamma\mathsf{TT}} t \equiv u : A \qquad \Gamma \mid (\mathtt{n}, \mathtt{false}) :: \ell \vdash_{\digamma\mathsf{TT}} t \equiv u : A}{\Gamma \mid \ell \vdash_{\digamma\mathsf{TT}} t \equiv u : A}\ \mathtt{n} \notin \mathrm{dom}(\ell)$$

Split Type Theory

Amazing new equations

$$\frac{}{\Gamma \mid \cdot \vdash_{\text{ʄTT}} (\texttt{if}\ ʄ\ 5\ \texttt{then}\ 0\ \texttt{else}\ 0) \equiv 0 : \mathbb{N}}$$

Split Type Theory

Amazing new equations

$$\frac{}{\Gamma \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (5, \texttt{true}) \vdash_{\mathfrak{f}\mathsf{TT}} (\texttt{if}\ \mathfrak{f}\ 5\ \texttt{then}\ 0\ \texttt{else}\ 0) \equiv 0 : \mathbb{N}}{\Gamma \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} (\texttt{if}\ \mathfrak{f}\ 5\ \texttt{then}\ 0\ \texttt{else}\ 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (5, \texttt{true}) \vdash_{\mathfrak{f}\mathsf{TT}} (\texttt{if}\ \texttt{true}\ \texttt{then}\ 0\ \texttt{else}\ 0) \equiv 0 : \mathbb{N}}{\Gamma \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} (\texttt{if}\ \mathfrak{f}\ 5\ \texttt{then}\ 0\ \texttt{else}\ 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (5, \texttt{true}) \vdash_{\mathfrak{f}\mathsf{TT}} 0 \equiv 0 : \mathbb{N}}{\Gamma \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} (\texttt{if}\ \mathfrak{f}\ 5\ \texttt{then}\ 0\ \texttt{else}\ 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (5, \mathtt{true}) \vdash_{\text{ʃTT}} 0 \equiv 0 : \mathbb{N} \qquad \Gamma \mid (5, \mathtt{false}) \vdash_{\text{ʃTT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 0) \equiv 0 : \mathbb{N}}{\Gamma \mid \cdot \vdash_{\text{ʃTT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (5, \mathtt{true}) \vdash_{\digamma\mathsf{TT}} 0 \equiv 0 : \mathbb{N} \quad \Gamma \mid (5, \mathtt{false}) \vdash_{\digamma\mathsf{TT}} (\mathtt{if}\ \mathtt{false}\ \mathtt{then}\ 0\ \mathtt{else}\ 0) \equiv 0 : \mathbb{N}}{\Gamma \mid \cdot \vdash_{\digamma\mathsf{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (5, \texttt{true}) \vdash_{\mathfrak{f}\mathsf{TT}} 0 \equiv 0 : \mathbb{N} \quad \Gamma \mid (5, \texttt{false}) \vdash_{\mathfrak{f}\mathsf{TT}} 0 \equiv 0 : \mathbb{N}}{\Gamma \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} (\texttt{if } \mathfrak{f}\ 5 \texttt{ then } 0 \texttt{ else } 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

Amazing new equations

$$\frac{\Gamma \mid (5, \texttt{true}) \vdash_{\mathfrak{f}\mathrm{TT}} 0 \equiv 0 : \mathbb{N} \qquad \Gamma \mid (5, \texttt{false}) \vdash_{\mathfrak{f}\mathrm{TT}} 0 \equiv 0 : \mathbb{N}}{\Gamma \mid \cdot \vdash_{\mathfrak{f}\mathrm{TT}} (\texttt{if } \mathfrak{f}\ 5 \texttt{ then } 0 \texttt{ else } 0) \equiv 0 : \mathbb{N}}\ 5 \notin \mathrm{dom}(\cdot)$$

Split Type Theory

What about Continuity?

Split Type Theory

Continuity from canonicity

Split Type Theory

Continuity from canonicity

Usual Canonicity Theorem

If

$$\cdot \vdash_{\mathsf{MLTT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps.

Split Type Theory

Continuity from canonicity

Usual Canonicity Theorem

If

$$\cdot \vdash_{\mathsf{MLTT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps.

Modified Canonicity Theorem for $\digamma\mathsf{TT}$

If

$$\cdot \mid \cdot \vdash_{\digamma\mathsf{TT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps **up to a tree of splits**.

Split Type Theory

Continuity from canonicity

Usual Canonicity Theorem

If

$$\cdot \vdash_{\mathsf{MLTT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps.

Modified Canonicity Theorem for $\digamma\mathsf{TT}$

If

$$\cdot \mid \cdot \vdash_{\digamma\mathsf{TT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps **up to a tree of splits**.

$$\cdot \mid \cdot \vdash_{\digamma\mathsf{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 1) : \mathbb{N}$$

Split Type Theory

Continuity from canonicity

Usual Canonicity Theorem

If

$$\cdot \vdash_{\mathsf{MLTT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps.

Modified Canonicity Theorem for $\mathfrak{f}\mathsf{TT}$

If

$$\cdot \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps **up to a tree of splits**.

$$5$$

$$\cdot \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} (\texttt{if } \mathfrak{f}\ 5 \texttt{ then } 0 \texttt{ else } 1) : \mathbb{N}$$

Split Type Theory

Continuity from canonicity

Usual Canonicity Theorem

If

$$\cdot \vdash_{\mathrm{MLTT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps.

Modified Canonicity Theorem for $\mathfrak{f}\mathrm{TT}$

If

$$\cdot \mid \cdot \vdash_{\mathfrak{f}\mathrm{TT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps **up to a tree of splits**.

5

$$\cdot \mid \cdot \vdash_{\mathfrak{f}\mathrm{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 1) : \mathbb{N}$$

$$\cdot \mid (5, \mathtt{true}) \vdash_{\mathfrak{f}\mathrm{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 1) \longrightarrow^{*} 0 : \mathbb{N}$$

Split Type Theory

Continuity from canonicity

Usual Canonicity Theorem

If

$$\cdot \vdash_{\mathsf{MLTT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps.

Modified Canonicity Theorem for $_{\mathfrak{f}}\mathsf{TT}$

If

$$\cdot \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps **up to a tree of splits**.

$$5$$

$$\cdot \mid \cdot \vdash_{\mathfrak{f}\mathsf{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 1) : \mathbb{N}$$

↙ ↘

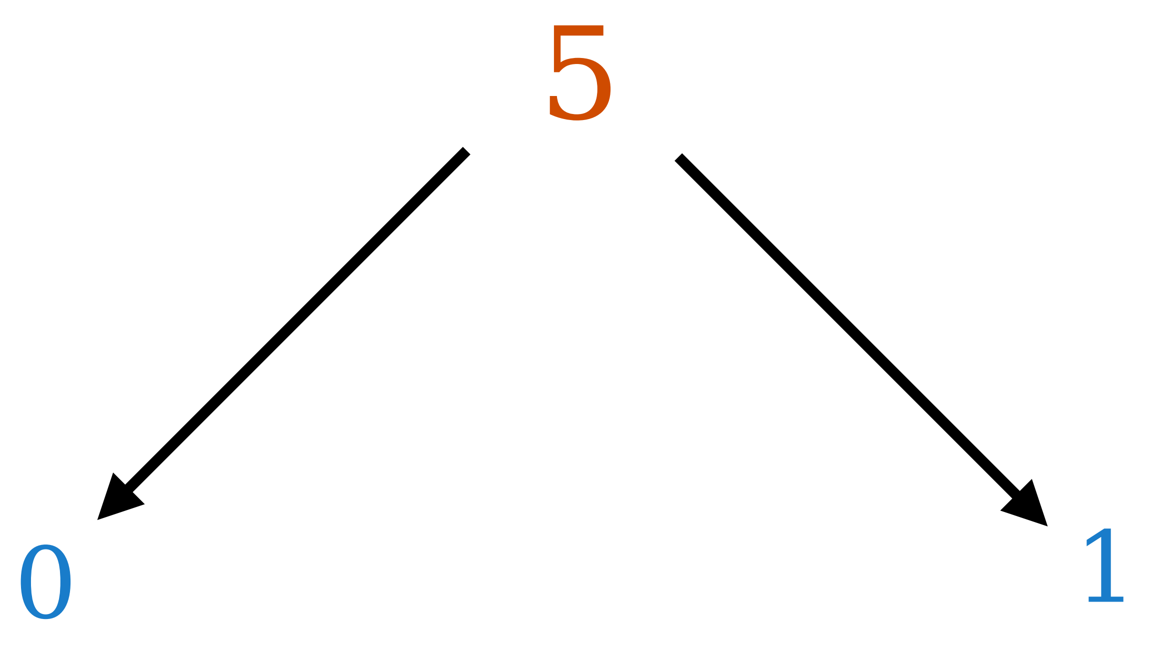
$$\cdot \mid (5, \mathtt{true}) \vdash_{\mathfrak{f}\mathsf{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 1) \longrightarrow^{*} 0 : \mathbb{N}$$

$$\cdot \mid (5, \mathtt{false}) \vdash_{\mathfrak{f}\mathsf{TT}} (\mathtt{if}\ \mathfrak{f}\ 5\ \mathtt{then}\ 0\ \mathtt{else}\ 1) \longrightarrow^{*} 1 : \mathbb{N}$$

Split Type Theory

Continuity from canonicity

$$\cdot \mid \cdot \vdash_{\digamma TT} (\texttt{if}\ \digamma\ 5\ \texttt{then}\ 0\ \texttt{else}\ 1) : \mathbb{N}$$



Usual Canonicity Theorem

If

$$\cdot \vdash_{\mathsf{MLTT}} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps.

Modified Canonicity Theorem for $\digamma$TT

If

$$\cdot \mid \cdot \vdash_{\digamma TT} t : \mathbb{N}$$

then t reduces to a numeral in a finite number of steps **up to a tree of splits**.

Normalizing Split TT

Normalizing Split TT

$$\Gamma \Vdash t \in A$$

Normalizing Split TT

$$\Gamma \Vdash t \in A$$

Strong reducibility

Normalizing Split TT

$$\Gamma \Vdash t \in A$$

Strong reducibility

Reducibility for the current ℓ.

Normalizing Split TT

$$\Gamma \Vdash t \in A$$

Strong reducibility

Reducibility for the current ℓ.

Weak reducibility

Normalizing Split TT

$$\Gamma \Vdash t \in A$$

Strong reducibility

Reducibility for the current ℓ.

Weak reducibility

Reducibility ”up to a tree of splits”

Normalizing Split TT

Strong reducibility

Normalizing Split TT

Strong reducibility

$$t \longrightarrow_{\ell}^{*} u$$

Normalizing Split TT

Strong reducibility

$$t \longrightarrow^*_\ell u$$

$$\frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow^*_\ell \overline{\mathrm{b}}}$$

Normalizing Split TT

Strong reducibility

$$t \longrightarrow_{\ell}^{*} u \qquad \frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow_{\ell}^{*} \overline{\mathrm{b}}}$$

$$\Gamma \mid \ell \Vdash^{s} A$$

Normalizing Split TT

Strong reducibility

$$t \longrightarrow^*_\ell u$$

$$\frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow^*_\ell \overline{\mathrm{b}}}$$

$$\Gamma \mid \ell \Vdash^{\mathrm{s}} A$$

$$\Gamma \mid \ell \Vdash^{\mathrm{s}} t \in A$$

Normalizing Split TT

Strong reducibility

$$t \longrightarrow^*_\ell u$$

$$\frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow^*_\ell \overline{\mathrm{b}}}$$

$$\Gamma \mid \ell \Vdash^{\mathrm{s}} A$$

$$\Gamma \mid \ell \Vdash^{\mathrm{s}} t \in A$$

$$\Gamma \mid \ell \Vdash^{\mathrm{s}} A \equiv B$$

Normalizing Split TT

Strong reducibility

$$t \longrightarrow^*_\ell u$$

$$\frac{(\mathsf{n}, \mathsf{b}) \in \ell}{\mathfrak{f}\ \overline{\mathsf{n}} \longrightarrow^*_\ell \overline{\mathsf{b}}}$$

$\Gamma \mid \ell \Vdash^{s} A$

$\Gamma \mid \ell \Vdash^{s} t \in A$

$\Gamma \mid \ell \Vdash^{s} A \equiv B$

$\Gamma \mid \ell \Vdash^{s} t \equiv u \in A$

Normalizing Split TT

Strong reducibility

$\Gamma \mid \ell \Vdash^{s} A$

$\Gamma \mid \ell \Vdash^{s} t \in A$

$\Gamma \mid \ell \Vdash^{s} A \equiv B$

$\Gamma \mid \ell \Vdash^{s} t \equiv u \in A$

$$t \longrightarrow_{\ell}^{*} u$$

$$\frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow_{\ell}^{*} \overline{\mathrm{b}}}$$

$$\Gamma \mid \ell \Vdash^{s} t \in \mathbb{B}$$

Normalizing Split TT

Strong reducibility

$\Gamma \mid \ell \Vdash^{s} A$

$\Gamma \mid \ell \Vdash^{s} t \in A$

$\Gamma \mid \ell \Vdash^{s} A \equiv B$

$\Gamma \mid \ell \Vdash^{s} t \equiv u \in A$

$$t \longrightarrow^{*}_{\ell} u$$

$$\frac{(\mathsf{n}, \mathsf{b}) \in \ell}{\mathfrak{f}\ \overline{\mathsf{n}} \longrightarrow^{*}_{\ell} \overline{\mathsf{b}}}$$

$$\Gamma \mid \ell \Vdash^{s} t \in \mathbb{B} \quad \longrightarrow$$

Normalizing Split TT

Strong reducibility

$\Gamma \mid \ell \Vdash^{s} A$

$\Gamma \mid \ell \Vdash^{s} t \in A$

$\Gamma \mid \ell \Vdash^{s} A \equiv B$

$\Gamma \mid \ell \Vdash^{s} t \equiv u \in A$

$t \longrightarrow^{*}_{\ell} u$

$$\frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow^{*}_{\ell} \overline{\mathrm{b}}}$$

$\Gamma \mid \ell \Vdash^{s} t \in \mathbb{B}$ ⟶ $t \longrightarrow^{*}_{\ell} \mathtt{true}$

Normalizing Split TT

Strong reducibility

$\Gamma \mid \ell \Vdash^{s} A$

$\Gamma \mid \ell \Vdash^{s} t \in A$

$\Gamma \mid \ell \Vdash^{s} A \equiv B$

$\Gamma \mid \ell \Vdash^{s} t \equiv u \in A$

$$t \longrightarrow^{*}_{\ell} u$$

$$\frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow^{*}_{\ell} \overline{\mathrm{b}}}$$

$$\Gamma \mid \ell \Vdash^{s} t \in \mathbb{B} \quad \Longrightarrow \quad t \longrightarrow^{*}_{\ell} \mathtt{true} \qquad t \longrightarrow^{*}_{\ell} \mathtt{false}$$

Normalizing Split TT

Strong reducibility

$\Gamma \mid \ell \Vdash^{s} A$

$\Gamma \mid \ell \Vdash^{s} t \in A$

$\Gamma \mid \ell \Vdash^{s} A \equiv B$

$\Gamma \mid \ell \Vdash^{s} t \equiv u \in A$

$t \longrightarrow_{\ell}^{*} u$

$$\frac{(\mathrm{n}, \mathrm{b}) \in \ell}{\mathfrak{f}\ \overline{\mathrm{n}} \longrightarrow_{\ell}^{*} \overline{\mathrm{b}}}$$

$$\Gamma \mid \ell \Vdash^{s} t \in \mathbb{B} \quad \Longrightarrow$$

$t \longrightarrow_{\ell}^{*} \texttt{true}$

$t \longrightarrow_{\ell}^{*} \texttt{false}$

$t \longrightarrow_{\ell}^{*} n$ $\qquad$ $\mathrm{ne}\ n$

$\Gamma \mid \ell \vdash n : \mathbb{B}$

Normalizing Split TT

Weak reducibility

Normalizing Split TT

Weak reducibility

$$\Gamma \mid \ell \Vdash^{\mathrm{w}} t \in A$$

Normalizing Split TT

Weak reducibility

$$\Gamma \mid \ell \Vdash^{\mathrm{w}} t \in A$$

$$\downarrow$$

Normalizing Split TT

Weak reducibility

$$\Gamma \mid \ell \Vdash^{\mathrm{w}} t \in A$$

$$\downarrow$$

m

Normalizing Split TT

Weak reducibility

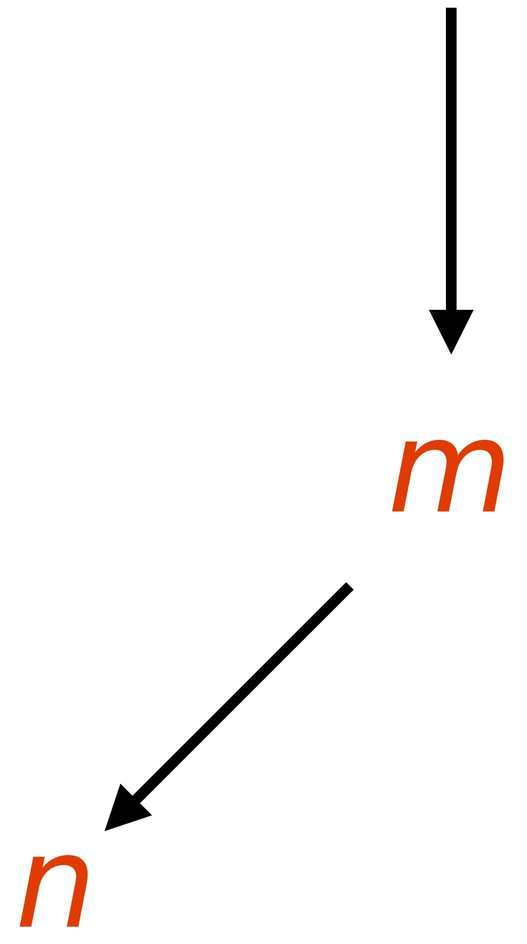
$$\Gamma \mid \ell \Vdash^{\mathrm{w}} t \in A$$

m

Normalizing Split TT

Weak reducibility

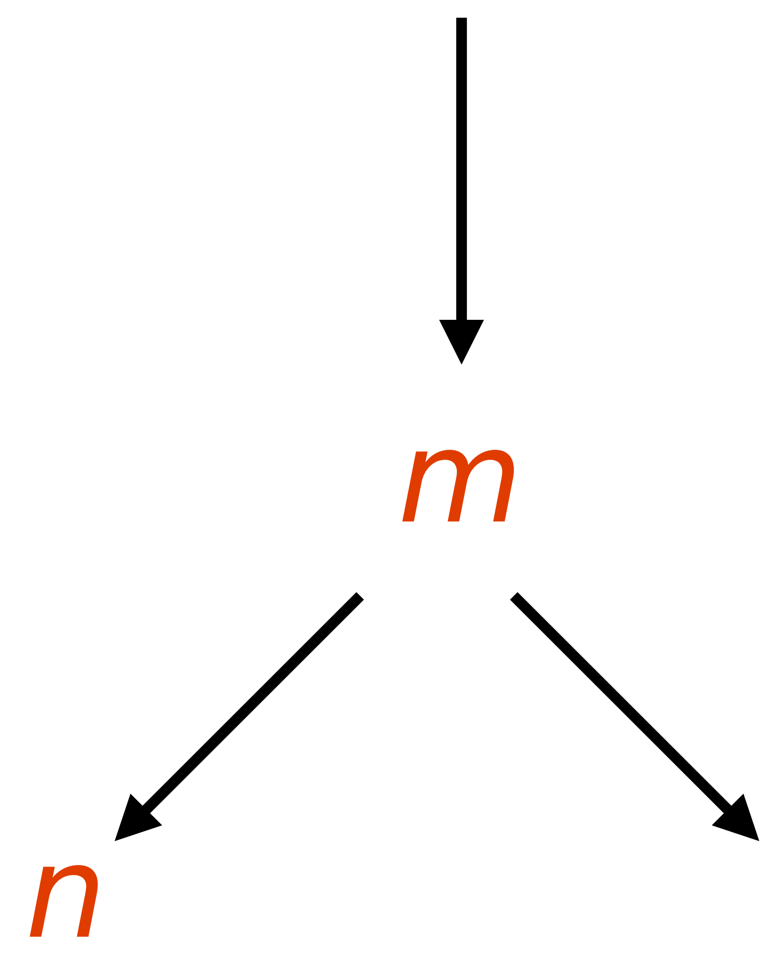
$$\Gamma \mid \ell \Vdash^{\mathrm{w}} t \in A$$



Normalizing Split TT

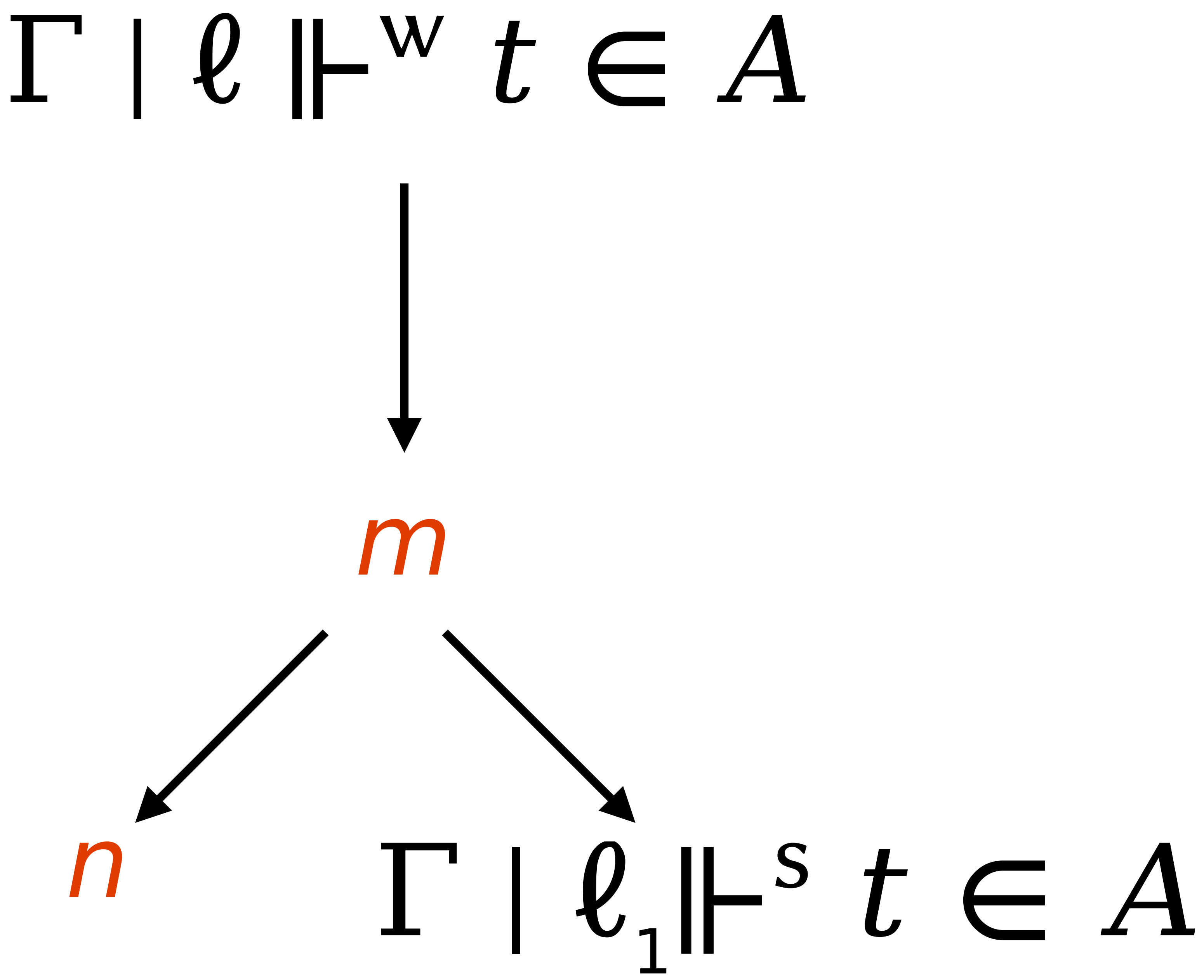
Weak reducibility

$$\Gamma \mid \ell \Vdash^{\mathrm{w}} t \in A$$



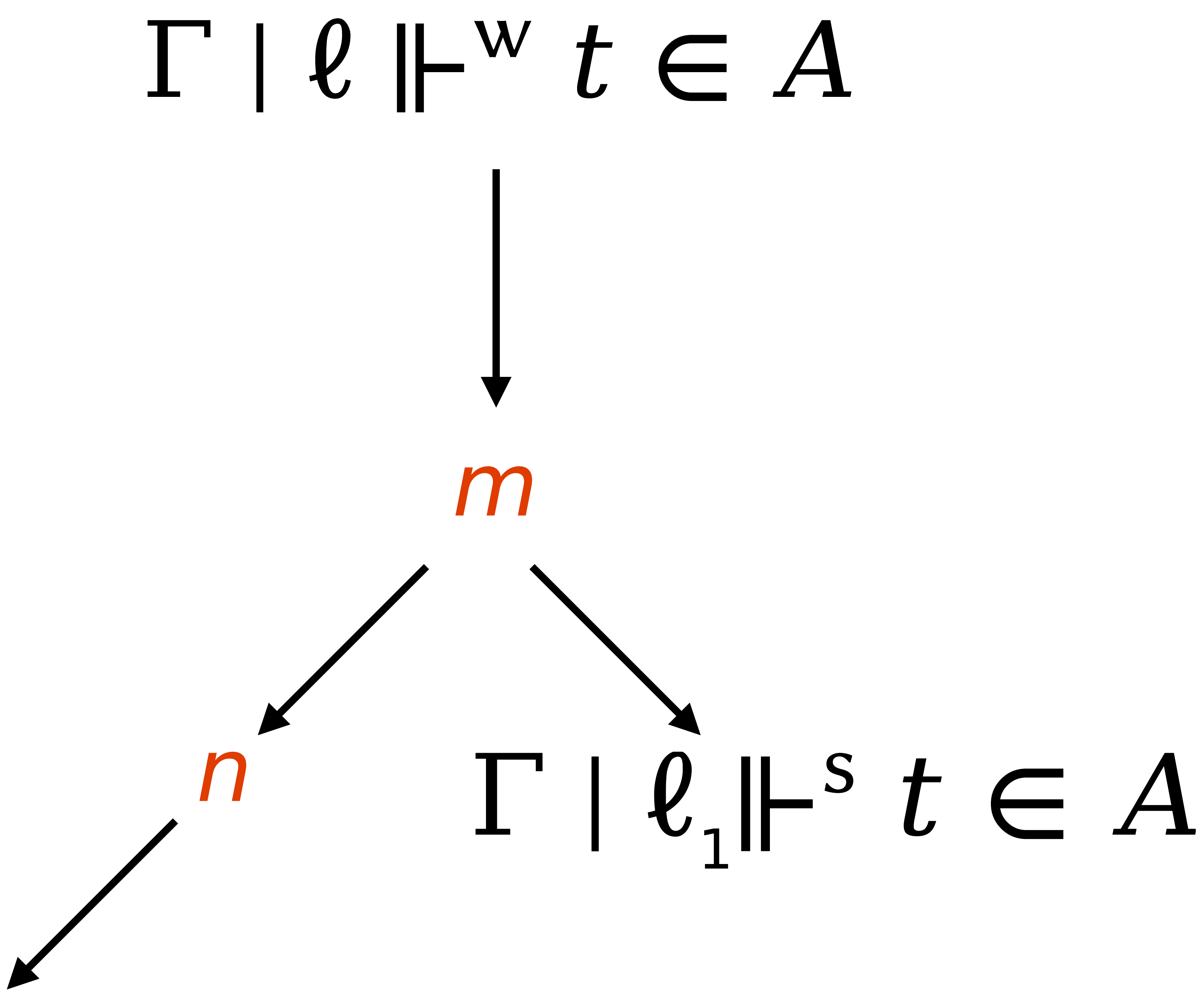
Normalizing Split TT

Weak reducibility



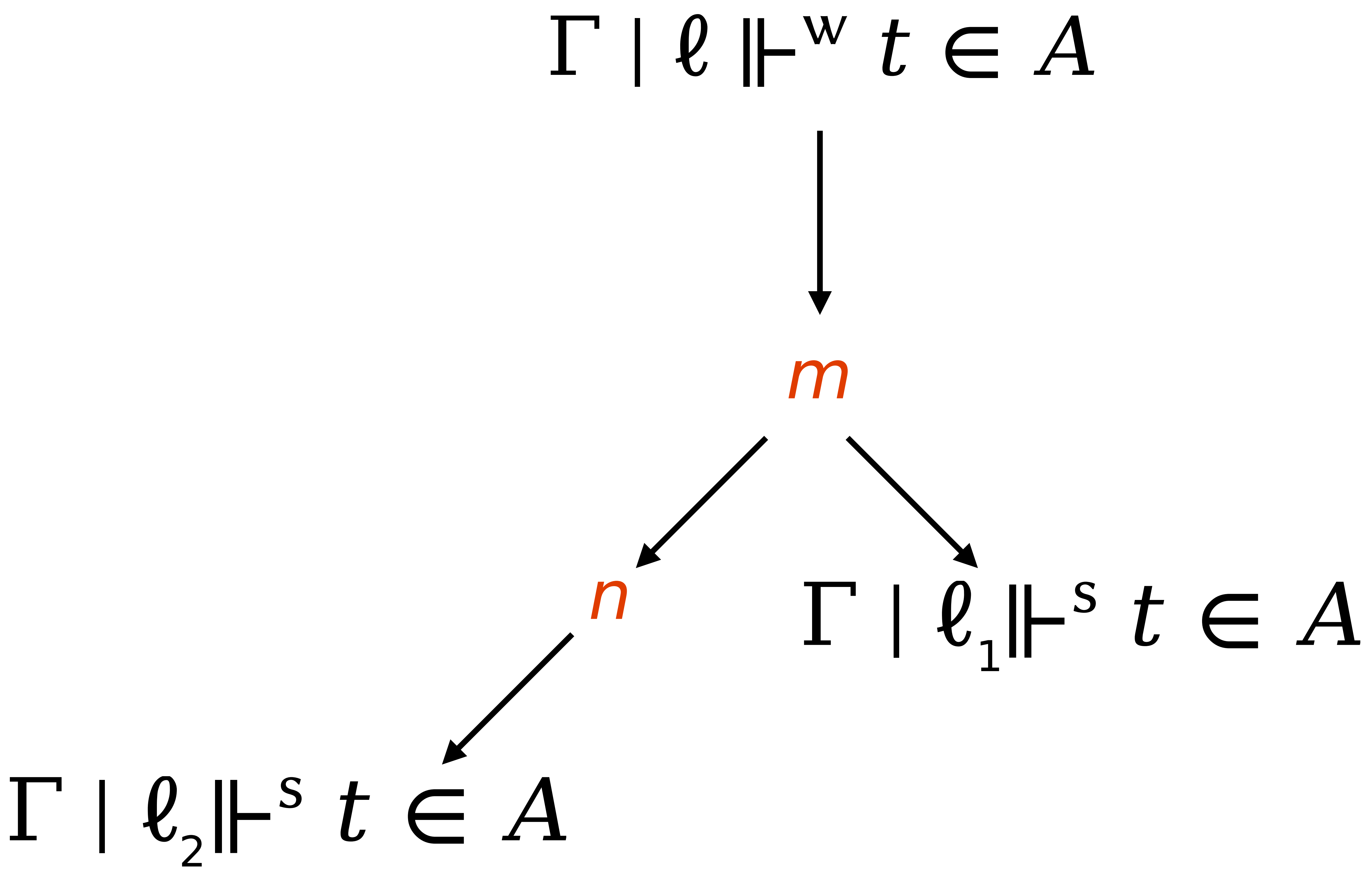
Normalizing Split TT

Weak reducibility



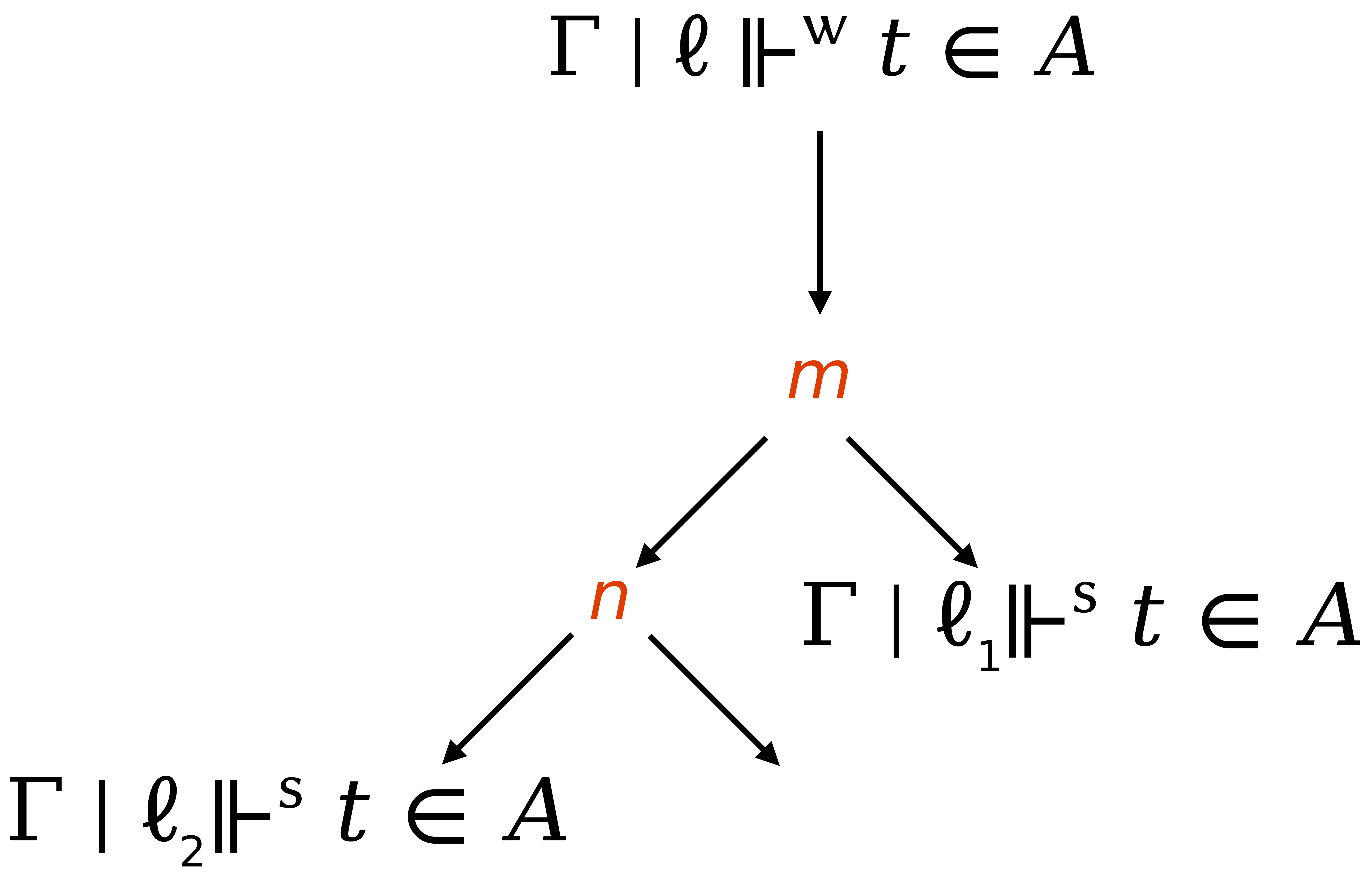
Normalizing Split TT

Weak reducibility



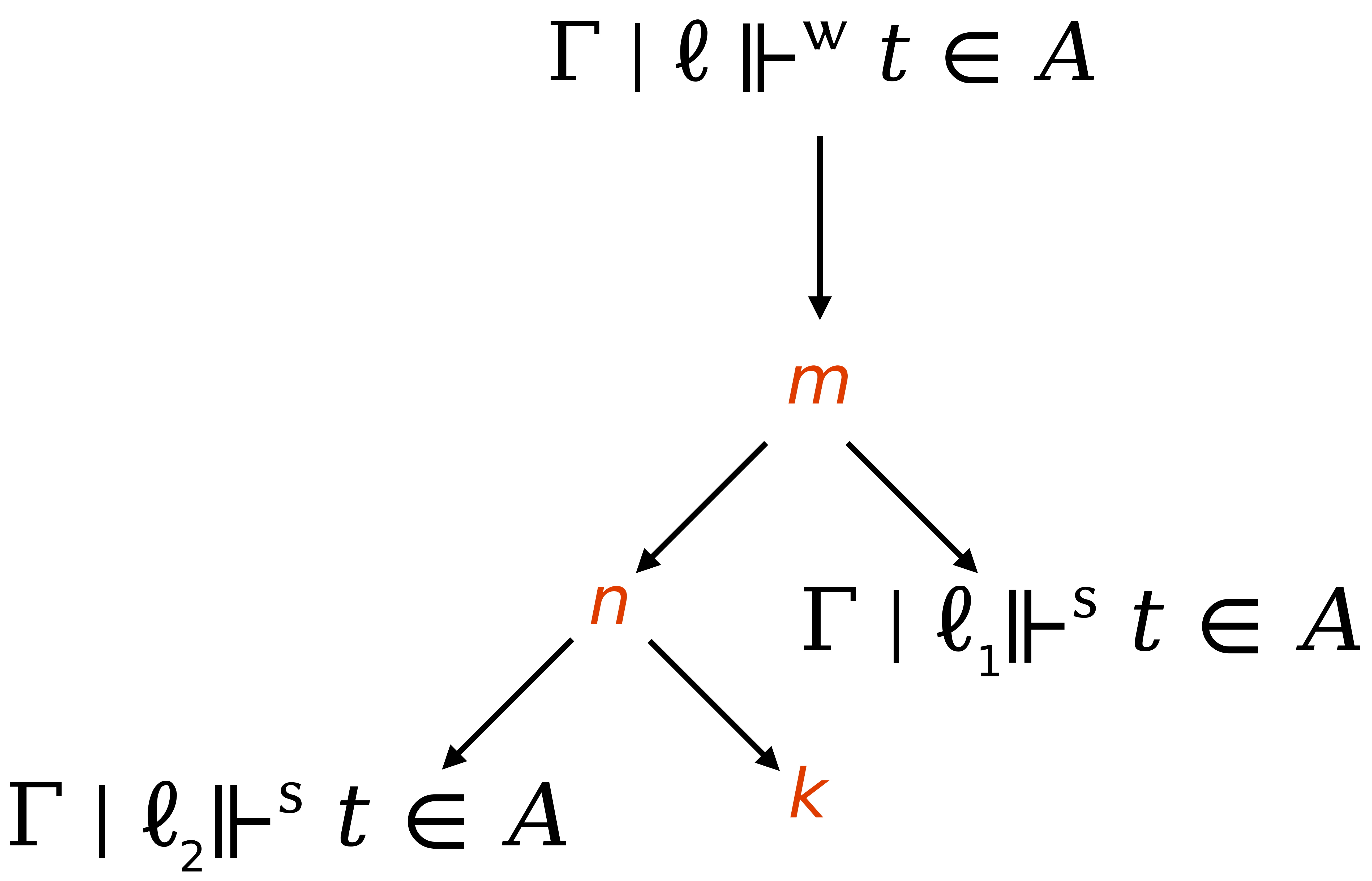
Normalizing Split TT

Weak reducibility



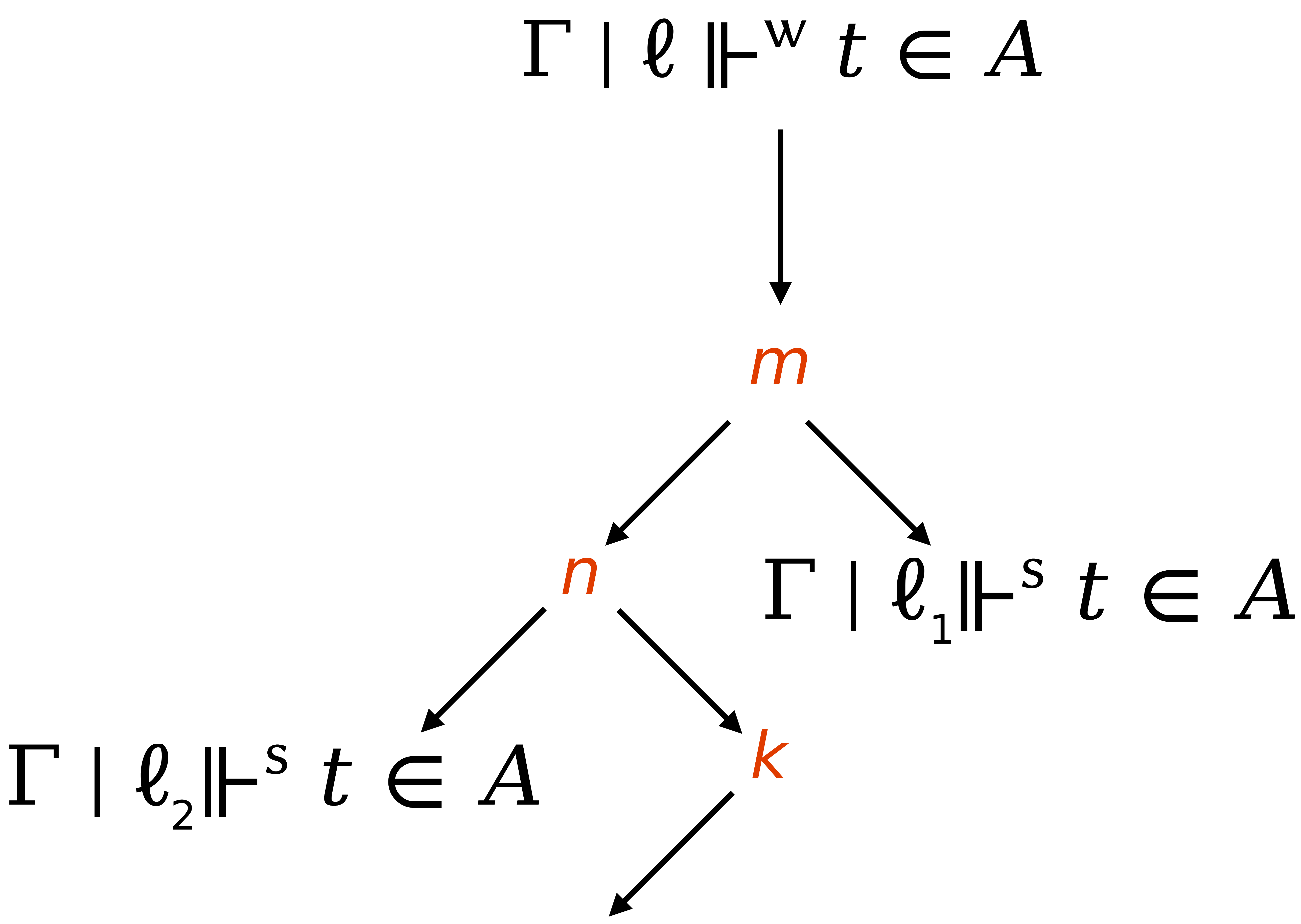
Normalizing Split TT

Weak reducibility



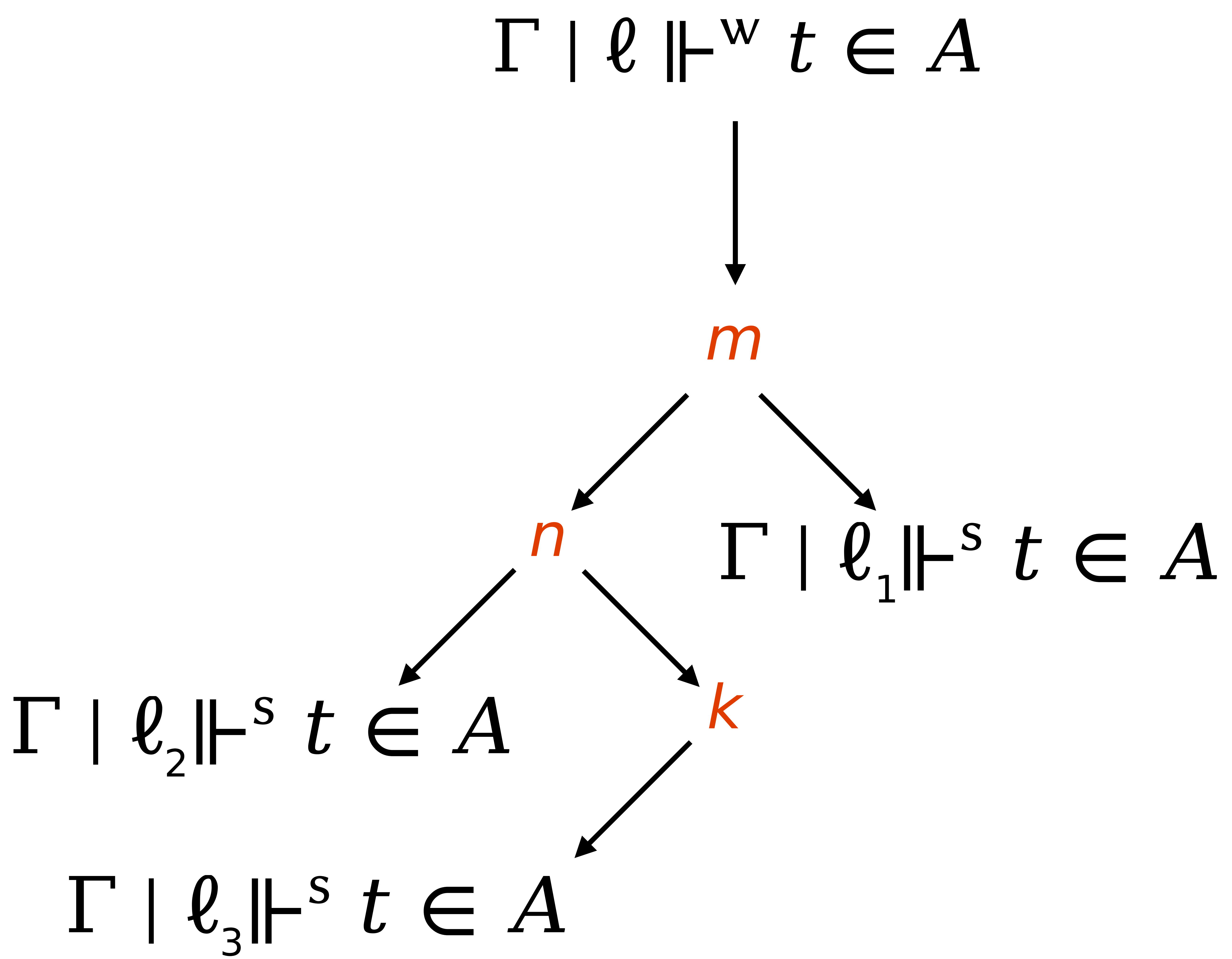
Normalizing Split TT

Weak reducibility



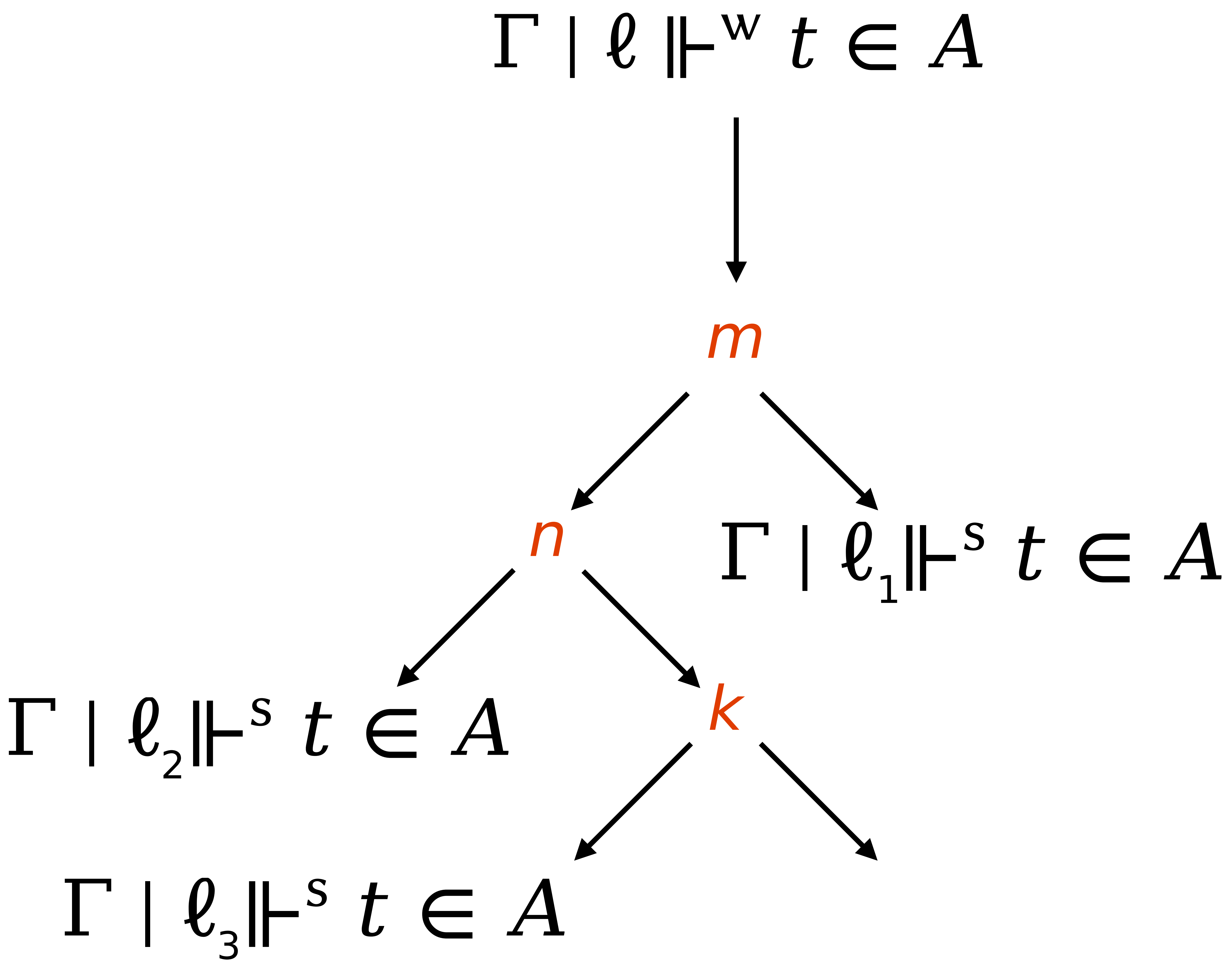
Normalizing Split TT

Weak reducibility



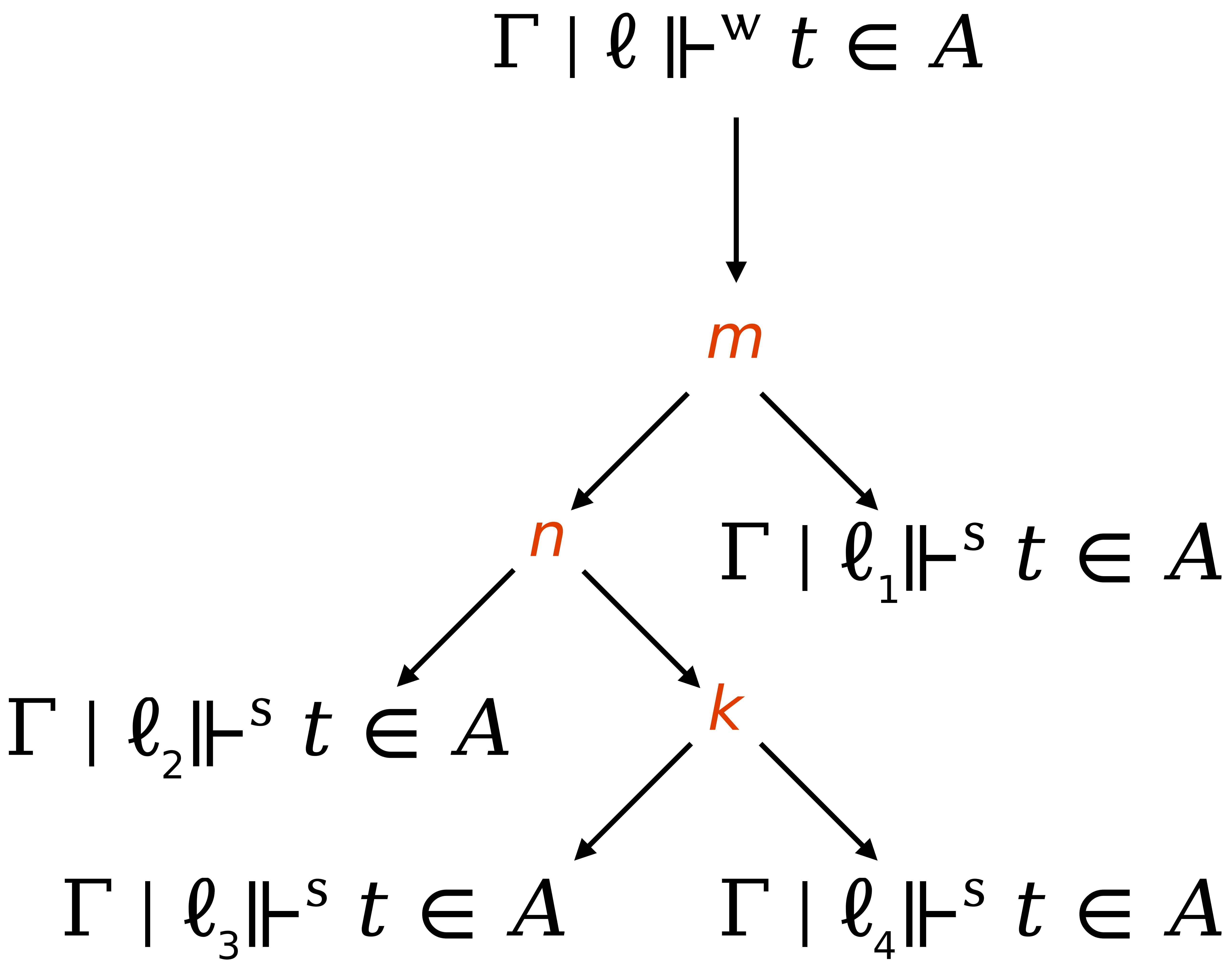
Normalizing Split TT

Weak reducibility



Normalizing Split TT

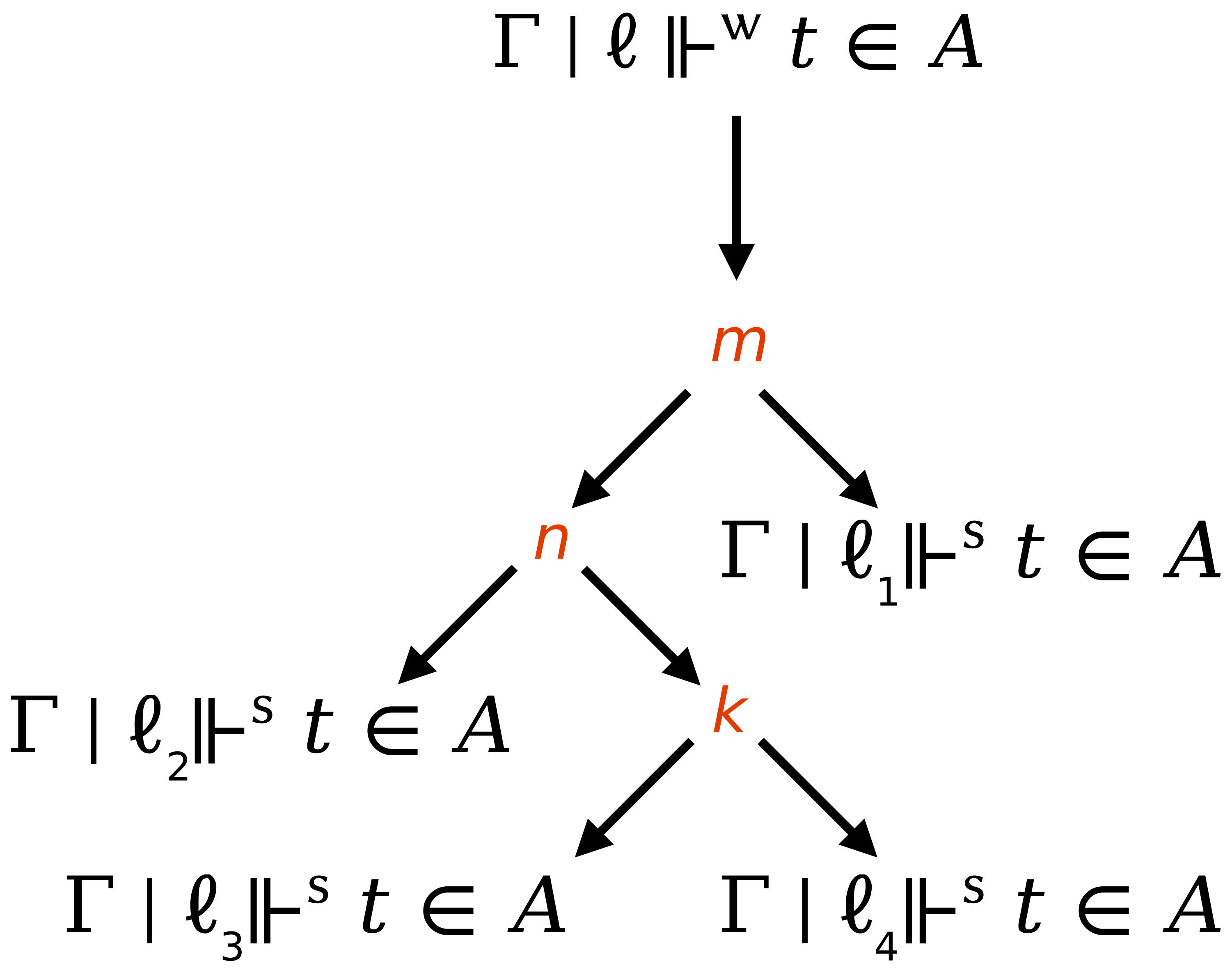
Weak reducibility



Normalizing Split TT

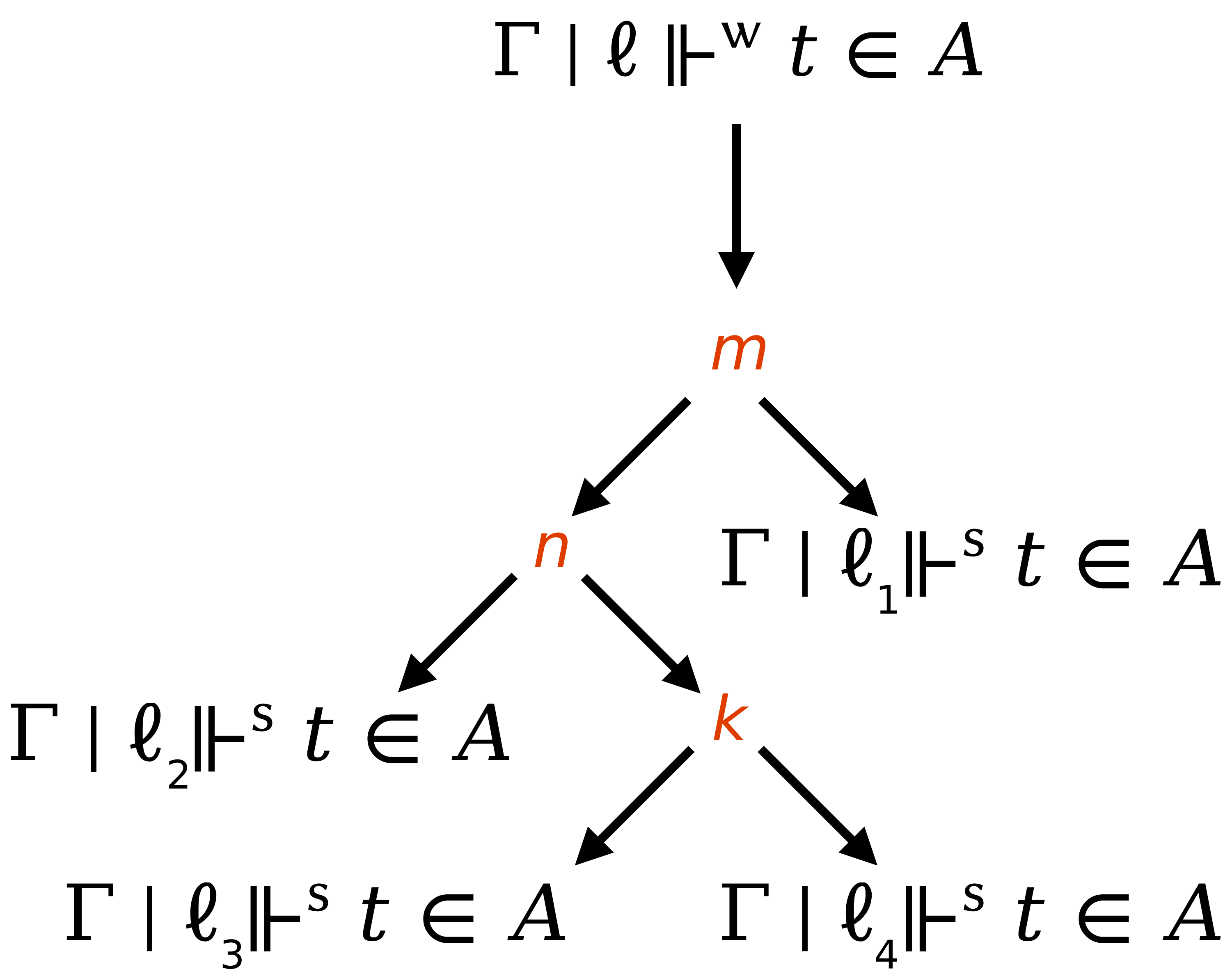
Weak reducibility

$$
\begin{array}{l}
\texttt{Inductive}\ \mathfrak{D}_{\ell} : \square \quad := \\
|\quad \eta_{\ell} : \mathfrak{D}_{\ell} \\
|\quad \beta_{\ell} : \Pi(\mathrm{n} \notin \mathrm{dom}(\ell)).\,(\Pi(b : \mathbb{B}).\,\mathfrak{D}_{(\mathrm{n},\mathrm{b})::\ell}) \to \mathfrak{D}_{\ell}.
\end{array}
$$



Normalizing Split TT

Weak reducibility



`Inductive` $\mathfrak{D}_\ell : \square \quad :=$

$| \quad \eta_\ell : \mathfrak{D}_\ell$

$| \quad \beta_\ell : \Pi(\mathrm{n} \notin \mathrm{dom}(\ell)).\,(\Pi(b : \mathbb{B}).\,\mathfrak{D}_{(\mathrm{n},\mathrm{b})::\ell}) \to \mathfrak{D}_\ell.$

$$\Gamma \mid \ell \Vdash^{\mathrm{w}} t \in A \coloneqq \Sigma d : \mathfrak{D}_\ell.\, \Pi \ell' \preceq d.\, \Gamma \mid \ell \Vdash^{\mathrm{s}} t \in A$$

Normalizing Split TT

Everything is normal

Martin-Löf *à la* Coq

Arthur Adjedj
ENS Paris Saclay, Université Paris-Saclay
Gif-sur-Yvette, France

Meven Lennon-Bertrand
University of Cambridge
Cambridge, United Kingdom

Kenji Maillard
Inria
Nantes, France

Pierre-Marie Pédrot
Inria
Nantes, France

Loïc Pujet
University of Stockholm
Stockholm, Sweden

Abstract

We present an extensive mechanization of the metatheory of Martin-Löf Type Theory (MLTT) in the Coq proof assistant. Our development builds on pre-existing work in Agda to show not only the decidability of conversion, but also the decidability of type checking, using an approach guided by bidirectional type checking. From our proof of decidability, we obtain a certified and executable type checker for a full-fledged version of MLTT with support for Π, Σ, ℕ, and **Id** types, and one universe. Our development does not rely on impredicativity, induction-recursion or any axiom beyond MLTT extended with indexed inductive types and a handful of predicative universes, thus narrowing the gap between the object theory and the metatheory to a mere difference in universes. Furthermore, our formalization choices are geared towards a modular development that relies on

checker is spent on establishing meta-theoretic properties, which are necessary to ensure termination of the type checker but have little to do with its concrete implementation.

Acknowledging this tension leads to two radically different approaches. On the one hand, one can simply postulate normalization, to better concentrate on the difficulties faced when certifying a realistic type-checker. The most ambitious project to date that follows this approach is MetaCoq [Sozeau, Anand, et al. 2020; Sozeau, Forster, et al. 2023], which formalizes a nearly complete fragment of Coq's type system and provides a certified type checker aiming for execution in a realistic context, after extraction. On the other hand, one can concentrate on normalization and decidability of conversion, which are the most difficult theoretical problems. The most advanced formalizations on that end are Abel, Öhman, et al. [2017] and Wieczorek and Biernacki

Thank you!