Expansion in a Calculus with Explicit Substitutions

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Motivation

- The λ-calculus assumes unlimited duplication and erasing of variables – lacks notion of cost
- In S. Alves and M. Florido "Structural Rules and Algebraic Properties of Intersection Types" (ICTAC 2022)
  - ACI-intersection types (associative, commutative and idempotent) $\Rightarrow$ Curry type system and relevant type systems
  - AC-intersection types $\Rightarrow$ affine and linear type systems
- Can we relate intersection type systems to calculi that explicitly track resource usage?

Resource Calculus

- Tracks resource usage explicitly

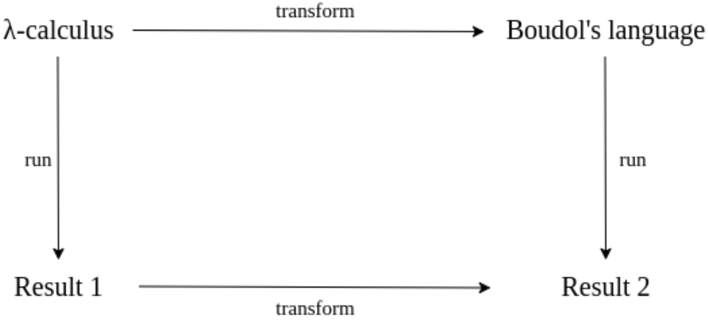
Example

$$\begin{array}{rcl} ((\lambda x.(xx^1))(\lambda z.z)^2) & \rightarrow & ((xx^1) < (\lambda z.z)^2/x >) \\ & \rightarrow & (((\lambda z.z)x^1) < (\lambda z.z)^1/x >) \\ & \rightarrow & ((z < x^1/z >) < (\lambda z.z)^1/x >) \\ & \rightarrow & ((x < 1/z >) < (\lambda z.z)^1/x >) \\ & \rightarrow & (((\lambda z.z) < 1/z >) < 1/x >) \\ & \equiv & \lambda z.z \end{array}$$

Gérard Boudol. "The lambda-calculus with multiplicities". In: *CONCUR'93: 4th International Conference on Concurrency Theory Hildesheim, Germany, August 23–26, 1993 Proceedings 4*. Springer. 1993, pp. 1–6

Goal

- Use a notion of expansion to transform λ-terms into terms that explicitly track resource usage (Boudol's calculus)



Mario Florido and Luis Damas. "Linearization of the lambda-calculus and its relation with intersection type systems". In: *Journal of Functional Programming* 14.5 (2004), pp. 519–546

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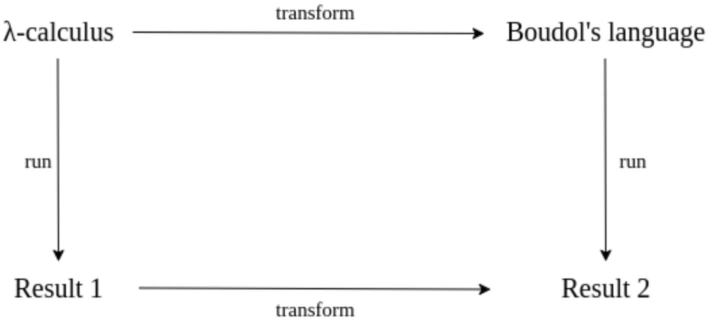

- Problem: In the λ-calculus, substitution is implicit in β-contraction

Explicit substitutions

- λx-calculus is an *explicit substitution calculus* that retains variable names instead of using indices *à la de Bruijn*
- Explicit substitution is given highest precedence
- It has explicit garbage collection

Kristoffer H Rose. *Explicit substitution: tutorial & survey*. University of Aarhus, 1996

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Example

$$
\begin{aligned}
(\lambda x.xx)(\lambda z.z) &\xrightarrow[\text{bxgc}]{} (xx) < x := \lambda z.z > \\
&\xrightarrow[\text{bxgc}]{} x < x := \lambda z.z > x < x := \lambda z.z > \\
&\xrightarrow[\text{bxgc}]{} (\lambda z.z)(\lambda z.z) \\
&\xrightarrow[\text{bxgc}]{} z < z := \lambda z.z > \\
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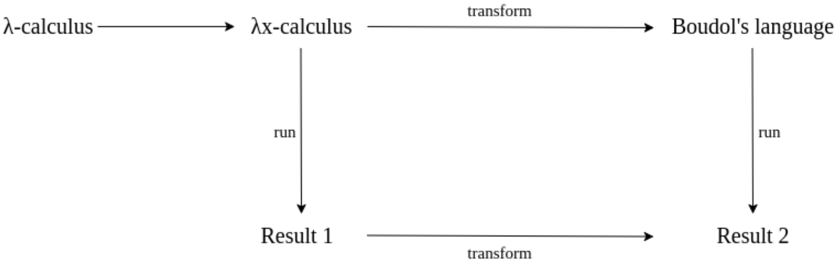
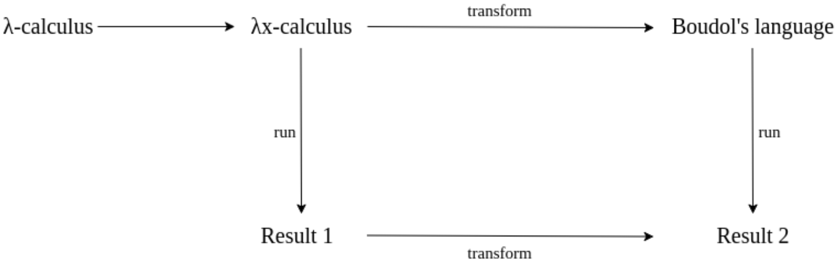

λ-calculus
λx-calculus
transform
Boudol's language
run
run
Result 1
transform
Result 2

Goal



- Can we relate terms typable by intersection types and each subset of Boudol's language?

Term expansion

- Focus on relating terms typable by ACI-intersection types with the subset of Boudol's language that deals only with infinite multiplicities (Λ^∞)
- The standard λ-calculus application MN, is denoted by (MN^∞), to indicate that the argument N is always available for function M

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Definition (Expansion)

Given a pair $M : \sigma$, where M is a λx-term and σ an intersection type, and a term N, we define a relation $\mathcal{E}(M : \sigma) \lhd N$, which we call *expansion*.

Example

Considering the term $(\lambda x.xx)(\lambda z.z)$, where $\sigma \equiv \alpha \rightarrow \alpha$

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$$\mathcal{E}(x : \sigma \to \sigma) \lhd x \text{ and } \mathcal{E}(x : \sigma) \lhd x$$

then $\mathcal{E}(xx : \sigma) \lhd (xx^{\infty})$
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Thus

$$\mathcal{E}((\lambda x.xx)(\lambda z.z) : \sigma) \lhd ((\lambda x.(xx^{\infty}))(\lambda z.z)^{\infty})$$

Expansion and Multiplicities

Theorem

Given a λx-term M and a type σ, such that $\mathcal{E}(M:\sigma) \lhd M^*$, if $M \underset{bxgc}{\twoheadrightarrow} V_1$ then $M^* \twoheadrightarrow_B V_2$ and $\mathcal{E}(V_1:\sigma) \lhd V_2$.

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Example

From the previous examples, we have $(\lambda x.xx)(\lambda z.z) \underset{\text{bxgc}}{\twoheadrightarrow} \lambda z.z$,

$$\mathcal{E}((\lambda x.xx)(\lambda z.z):\sigma) \lhd ((\lambda x.(xx^\infty))(\lambda z.z)^\infty)$$

and

$$\begin{array}{rcl} ((\lambda x.(xx^\infty))(\lambda z.z)^\infty) & \twoheadrightarrow & (((\lambda z.z) < x^\infty/z >) < (\lambda z.z)^\infty/x >) \\ & \equiv & \lambda z.z \end{array}$$

We also know that $\mathcal{E}(\lambda z.z : (\sigma \to \sigma) \cap \sigma) \lhd \lambda z.z$

Final remarks

- We proved that ACI-intersection types $\Rightarrow$ a resource calculus that deals with infinite multiplicities
- This serves as preliminary work towards proving AC-intersection types $\Rightarrow$ a resource calculus of finite multiplicities
- We also wish to look into an extension that deals with α-conversion

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Thank you!