Representing type theories in two-level type theory

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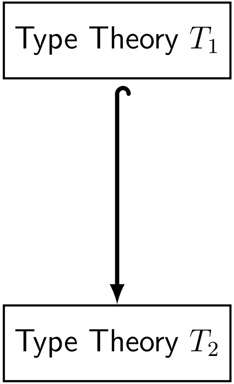
Goal: Understanding Extensions



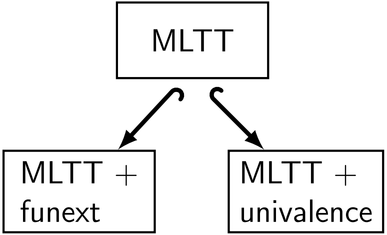
Situation:

- We understand T_1.
- We want to understand T_2.
- We want to compare T_2 to another extension of T_1.

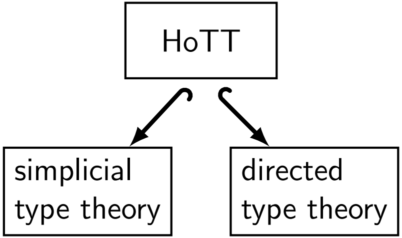
Two kinds of extensions:

- axiomatic extensions (easier to understand, can use proof assistants for T_1)
- structural extensions (trickier)

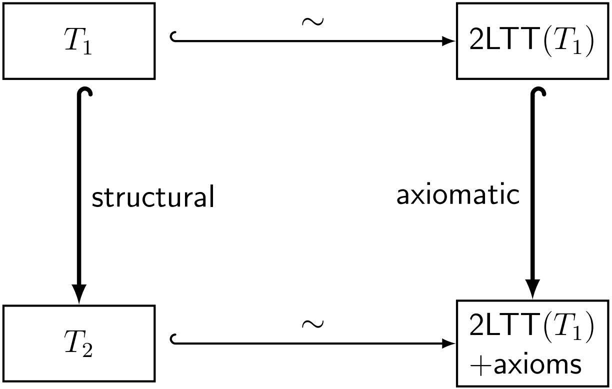
Axiomatic extensions:



Structural extensions:



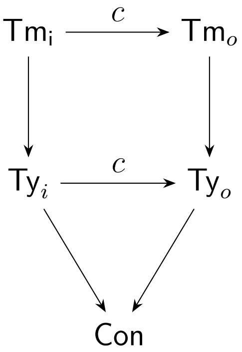
Approach: Make Extensions Axiomatic

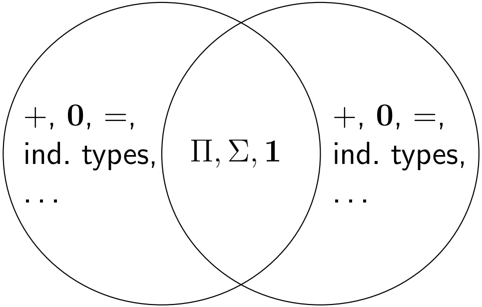


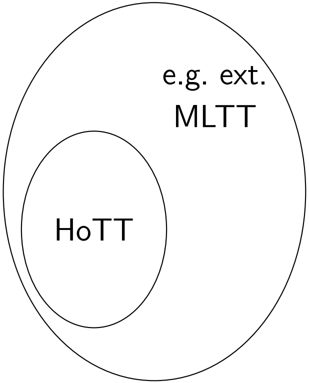
- We want to use two-level type theory (2LTT) to represent other theories.
- The vertical extensions are ideally conservative (i.e. don't change what is provable).
- This may lose computational properties.

Two-level type theory [Voe13; Cap17; ACK18; Ann+23]

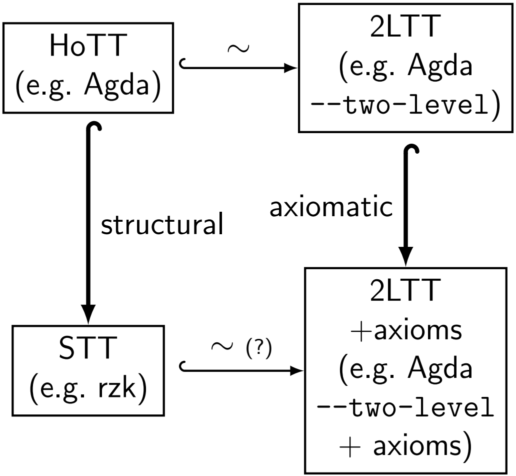
- Two type theories (*inner/fibrant* and *outer/exo* theory)
- Inner theory is the type theory of interest; outer theory is "just" auxiliary language







Example: simplicial type theory in 2LTT



- Riehl and Shulman's *simplicial type theory (STT)* [RS17] is HoTT with two additional components:
  - additional context layers to talk about *shape inclusions* and *extension types*;
  - simplicial shapes.
- Remainder of this talk: model STT in 2LTT.

Example: simplicial type theory in 2LTT; extension types

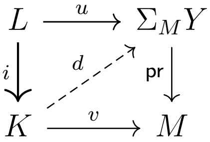
In simplicial TT, assume:

- $\Phi \subset \Psi$ are shapes (defined using the new context layers)
- A is a type on the "big" shape Ψ
- a is a term of A on the "small" shape Φ

Then: $\left\langle \Pi_{t:\Psi} A(t) \middle|^{\Phi}_{a} \right\rangle$ is the type of extensions of a.

In 2-level type theory:

- $i : L \to K$ is *cofibration* if $f \widehat{\pitchfork} \_$ preserves [trivial] fibrations.
- This means: For any fibrant family $Y : M \to \mathcal{U}$ and strictly commuting squares the type of (d) is fibrant (and contractible if Y is).



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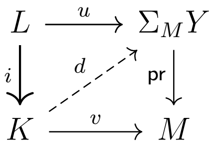
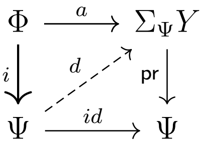


- Special case:



Shape inclusions of simplicial type theory = cofibrations of 2LTT
Extension types of simplicial type theory = properties of cofibrations

Example: simplicial type theory in 2LTT; simplicial shapes

Second ingredient of simplicial type theory: a directed interval.

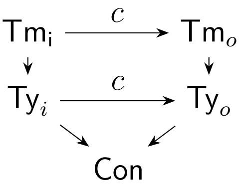
In simplicial type theory:

- Assume a bounded total order
  $(I, \leq, \bot, \top)$
  (or a variation)
- I is a 1-simplex ("line"),
  $\Delta^1 := I$
- Other simplicial shapes can be constructed, e.g.
  $\Delta^2 := \{(t_1, t_2) : I \times I \mid t_2 \leq t_1\}$

In 2-level type theory:

- We can mirror the STT approach and add cofibrancy assumptions.
- Alternatively:
  On the outer level, define S to be the subcategory of simplicial sets, spanned by subfunctors of representables
  (these are the "shapes of interest");
  then, assume a functor shape : $S \to \mathsf{U}^{\mathsf{strict}}$ that sends monos to cofibrations.

Instantiating 2-level type theory



We have assumed that the inner type theory is HoTT.
We have no requirements (yet) on the outer type theory.

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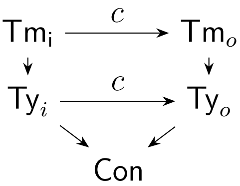

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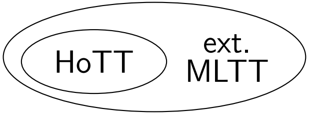
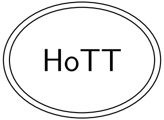
Possibility 1: Outer theory is ext. MLTT.



- Close to original approach [RS17]
- Less data
- Slightly more general

Possibility 2: Outer type theory is HoTT; conversion is id



- Purely in HoTT
- Matches [GWB24; GWB25]

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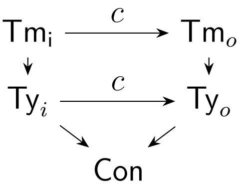

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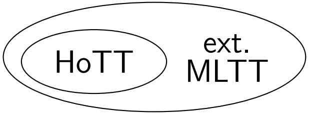

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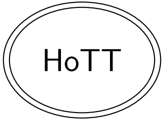

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Approaches "equivalent" b/c, for $\Sigma_X Y \to X$ and $x_0 : X$, we have:
strict fibre $\equiv Y(x_0) \simeq \Sigma(x : X).Y(x) \times (x = x_0) \equiv$ homotopy fibre

References I

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[GWB24] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. "Directed univalence in simplicial homotopy type theory". arXiv: 2407.09146. 2024.

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[RS17] Emily Riehl and Michael Shulman. "A type theory for synthetic ∞-categories". In: *Higher Structures* 1.1 (2017), pp. 147–224. DOI: 10.21136/hs.2017.06.

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