Irregular models of type theory

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Question (Escardó)

Are there models of type theory that do not have propositional truncation?

Alternatively: Is there a clever way to construct propositional truncation from other type constructors and univalence, or is this impossible?

Theorem (S)

There are locally cartesian closed categories with initial object, finite disjoint products, W-types and an infinite cumulative sequence of universes, that are not regular (i.e. do not have propositional truncation).

Theorem (S)

There are models of extensional and univalent type theory with a cumulative sequence of universes such that

1. *The model has an empty type and disjoint products.*
2. *The model has W-types.*
3. *The model has a circle type $\mathbb{S}^1$ (probably!)*
4. *There is a type in the lowest universe $\mathcal{U}_0$ that does not have a propositional truncation in any universe.*

We will construct an example using *Lifschitz realizability*. Originally due to Lifschitz. This presentation is based on variants due to Lee and Van Oosten. See also later work of Rathjen and Swan, Koutsoulis.

1. Using realizability over $\mathcal{K}_1$ we get a model of HoTT and the axioms:
   1.1 Computable choice: Given a $\neg\neg$-stable relation $R : \mathbb{N} \times \mathbb{N} \to \mathrm{hProp}_{\neg\neg}$ with $\neg\neg$-stable domain (i.e. for all n, $\neg\neg\exists m\, R(n, m) \to \exists m\, R(n, m)$), there is a partial computable function φ_e such that if $R(n, m)$ for any m, then $R(n, \varphi_e(n))$.
   1.2 Every $\neg\neg$-well founded relation on every set X is well-founded.
2. We construct a sequence of reflective subuniverses $\mathcal{L}_2 \hookrightarrow \mathcal{L}_3 \hookrightarrow \ldots \hookrightarrow \mathcal{U}$ where
   2.1 Each inclusion preserves the empty type, coproducts and W-types.
   2.2 No inclusion preserves propositional truncation.

Definition

Write $\mathbb{N}_\infty$ to be the set of decreasing binary sequences of natural numbers. It has a bounded lattice structure $\top, \bot, \vee, \wedge$ defined pointwise.

Definition

The axiom $\mathbf{LLPO}_n$ states that if $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_\infty$ are such that $\alpha_i \vee \alpha_j = \top$ for all $i \neq j$, then $\alpha_i = \top$ for some i.

Definition

For each n we define $\bigcirc_n$ to be the nullification modality of the propositions $\| \sum_i \alpha_i = \top \|$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_\infty$ are such that $\alpha_i \vee \alpha_j = \top$ for $i \neq j$ and $\mathcal{L}_n$ the corresponding reflective subuniverse of $\mathcal{U}$.

By construction, a type A belongs to $\mathcal{L}_n$ when it is null for the generating propositions:

$$\begin{array}{ccc} \| \sum_i \alpha_i = \top \| & \longrightarrow & A \\ \downarrow & \overset{!}{\nearrow} & \\ 1 & & \end{array}$$

Lemma

Each generator of $\bigcirc_n$ *is* $\bigcirc_{n+1}$*-modal.*

Proof.

We have a map $\| \sum_{1 \le i \le n+1} \alpha_i = \top \| \to \| \sum_{1 \le j \le n} \beta_j = \top \|$. By computable choice, this gives us a map $f : \mathsf{Fin}_{n+1} \to \mathsf{Fin}_n$ such that if $\alpha_i = \top$ then $\beta_{f(i)} = \top$. By the pigeonhole principle we have $i \neq i'$ such that $f(i) = f(i')$. From $\alpha_i \vee \alpha_{i'} = \top$ we can deduce $\beta_{f(i)} = \top$. □

Lemma

Each modality $\bigcirc_n$ *is non trivial.*

Proof.

Assume the generators of $\bigcirc_n$ are already contractible. Given $e \in \mathbb{N}$ and $1 \le i \le n$ define $\alpha_{e,i}(k)$ to be 0 if $\varphi_e(e)$ halts within k steps and $\varphi_e(e) \equiv i \mod n$, and otherwise 1. By the contractibility assumption and computable choice we can choose $f(i)$ computably such that $\alpha_{e,f(e)} = \top$. We get a contradiction by diagonalisation. □

Theorem

Each $\mathcal{L}_n$ is closed under coproducts and W-types.

Theorem

If we additionally assume that every proposition $\alpha = \top$ is projective in our base realizability model, then $\mathbb{S}^1$ belongs to $\mathcal{L}_2$, and so to all $\mathcal{L}_n$.

(This holds in extensional realizability models such as assemblies and the effective topos, and Should Probably Hold™ in the new model of HoTT due to Sattler.)

We can use the lemmas to show the required results that inclusions $\mathcal{L}_n \hookrightarrow \mathcal{L}_{n+1}$ preserve "everything except truncation."

To get a non regular lcc with disjoint sums and W-types, we can simply take the colimit. Intuitively: the countable union $\bigcup_n \mathcal{L}_n$. We then take the slice category over all finite sequences $\alpha_1, \ldots, \alpha_n$ where $\alpha_i \vee \alpha_j = \top$ for $i \neq j$. Then the truncation $\| \sum_{1 \leq i \leq n} \alpha_i = \top \|$ belongs to $\mathcal{L}_{n+1}$, but *not* $\mathcal{L}_n$. Hence there is no one universe that contains all such truncations.

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There is a problem defining universes in this model: If we try to define a single universe as a countable union of all $\mathcal{L}_n$, it will not be closed under $\sum$-types. If we define the nth universe as $\mathcal{L}_n$ we have another problem: it still satisfies a weak form of truncation. Every type $A : \mathcal{L}_n$ has a truncation $\|A\|$ in the next universe $\mathcal{L}_{n+1}$ and in $\mathcal{L}_n$ it has a "local" truncation $\bigcirc_n \|A\|$ that eliminates into propositions in $\mathcal{L}_n$.

Theorem (Shulman)

If $\mathcal{E}$ is a model of univalent type theory, then so is $\mathcal{E}^{\to}$.

Given a pair of univalent universes $\mathcal{U}, \mathcal{V}$

$$\sum_{A:\mathcal{U}} \mathcal{V}^A \longrightarrow \mathcal{U}$$

If $\mathcal{U}$ is a subuniverse of $\mathcal{V}$, we can instead view the inclusion $\mathcal{U} \hookrightarrow \mathcal{V}$ as an object of $\mathcal{E}^{\to}$ and then as a univalent universe. This is equivalent to restricting to the subtype of those pairs $A : \mathcal{U}, B : \mathcal{V}^A$ in the Shulman universe where $A : \mathcal{V}$ and each $B(a)$ is contractible. If the inclusion does not preserve propositional truncation, then as a universe in $\mathcal{E}^{\to}$ it does not have truncation.

Still to do:

1. More direct interpretation for $\mathbb{S}^1$
2. Other HITs?
3. Other applications.

Thanks for you attention!