TYPE THEORY & THEMES IN PHILOSOPHICAL LOGIC

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https://consequently.org/p/2025/tt-tpl

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3 INTENSIONALITY & IDENTITY

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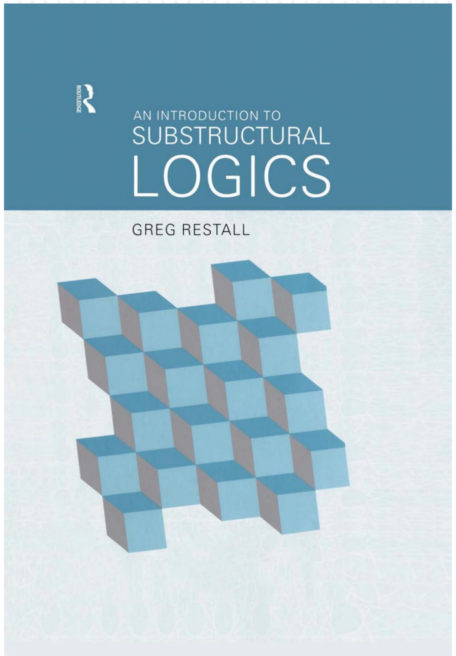
I work in PHILOSOPHICAL LOGIC



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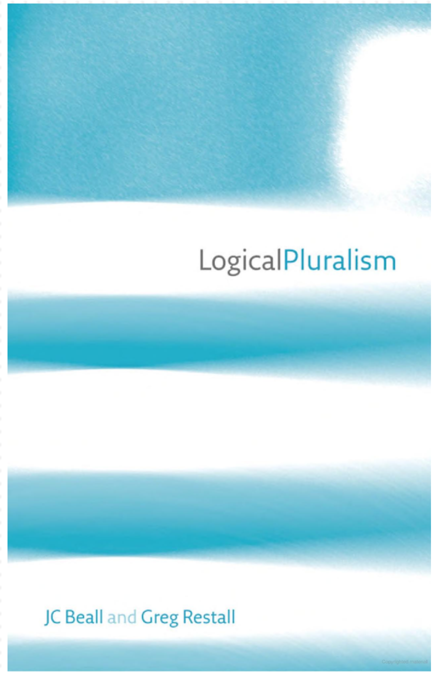

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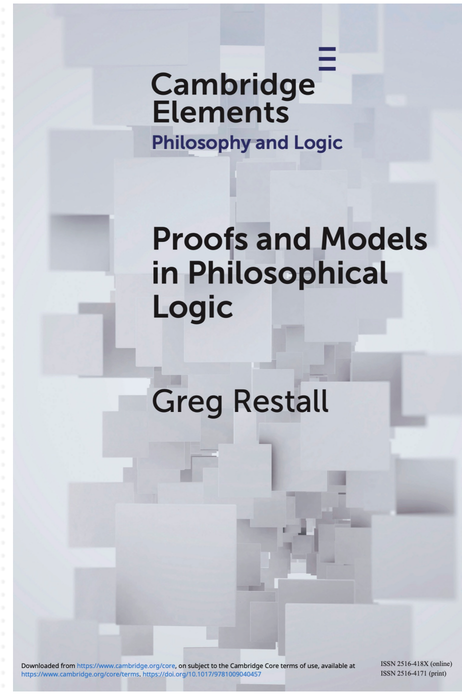

2022

I work to understand the connections between different techniques, traditions & approaches in logic & philosophy

TYPE THEORY is an exciting world I am beginning to explore.

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MODAL LOGICS — possibility & necessity; reasoning over times, ...

- Massive industry
  - "metaphysical necessity"
  - Epistemic logics
  - Montague style type theory in linguistics

Central tool: Kripke-style "possible worlds semantics."

Good tools at the level of types, not terms.
(Kripke models represent what follows from what — not why.)

Minority tradition ~ algebras & proof theory for modal logics.

RESIDUATION
Galois Connection

$$\frac{\overleftarrow{\Diamond} a \leq b}{a \leq \overrightarrow{\Box} b}$$

$\overleftarrow{\Diamond}$ some time in the past

$\overrightarrow{\Box}$ all times in the future

Generalises more naturally to categories & so, to type theoretical interpretation.

SUBSTRUCTURAL LOGICS — resources, relevance, paradox, syntax

$$\frac{a \otimes b \leq c}{a \leq b \to c}$$

"Substructural" since the standard structural rules of contraction, weakening, permutation, & even associativity may be absent.

Kripke models for modal logics generalise to the substructural setting.

$\vec{\Box}$ $\overleftarrow{\Diamond}$ — unary connective, binary relation

$\vec{\Box}$ universal forward
$\overleftarrow{\Diamond}$ existential backward

$\to$ $\otimes$ — binary connective, ternary relation

$\to$ universal forward
$\otimes$ existential backward

These models extend distributive lattices with $\to$, $\otimes$.

(Algebras, Coherence Spaces & Phase space models give natural non-distributive structures & categories.)

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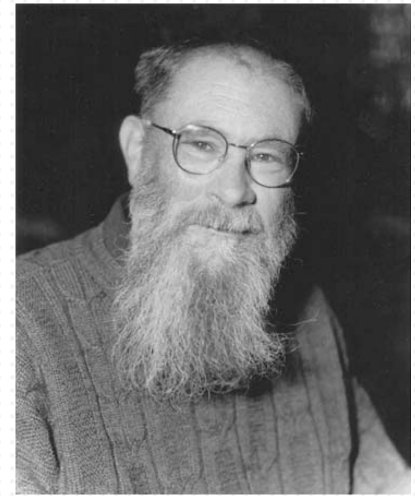

Identity is utterly simple and unproblematic.
Everything is identical to itself;
nothing is ever identical to anything else except itself.

— David LEWIS
On the Plurality of Worlds

- This is correct, but it is not the end of questions about identity

IDENTITY &
- NECESSITY
- PROOF
- CONSTRUCTION
- ISOMORPHISM

And philosophers have worked on these issues for a long time.

IDENTITY & NECESSITY

8 = the number of planets ✓

It is necessary that 8 = 8 ✓

It is necessary that the number of planets = 8 ✗

$\Box$[(The n where n = # planets) $n = 8$] ✗ — de dicto

(The n where n = # planets) $\Box$[$n = 8$] ✓ — de re

Scope makes a difference

IDENTITY & KNOWLEDGE / PROOF

Clark Kent = Superman. ✓

Lois Lane knows that Clark Kent is Clark Kent. ✓

Lois Lane knows that Clark Kent is Superman. ?

$f(x) = y$ ✓

Lois Lane shows that $y = y$ ✓

Lois Lane shows that $f(x) = y$?

$s = t$'s' & 't' have the same referent (value)

they might not have the same sense.

ISOMORPHISM & IDENTITY

Mathematical Structures — when are G_1 & G_2 the same group?

What is the relationship between isomorphism & identity?

(this is a part of deciding what mathematical structure is.)

In philosophy of mathematics this is explored in STRUCTURALISM, which is congenial to category theoretical (& HoTT, Cubical) presentation, but these views are not identical.

(See, especially, Colin Maclarty, Steve Awodey.)

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PROOF THEORY ♥ CONSTRUCTIVE LOGIC

Gentzen, Heyting, Dummett, PML, Prawitz, Girard

Understanding the Classical / Constructive boundary is an active research area in many directions.

Translations: Classical $\xhookrightarrow{\text{DN}/\ldots}$ Constructive

Constructive $\xhookrightarrow[\text{Topological}]{\text{S4}}$ Classical Modal

The Context of Deduction:

What is the difference between $\Gamma \vdash A$ & $\Gamma \vdash A, \Delta$; $\Gamma, A \vdash \Delta$?

BILATERALISM: assertion & denial treated equally

A (MILDLY) BILATERAL PROOF OF PEIRCE'S LAW

$$
\dfrac{\dfrac{\dfrac{\dfrac{[(A\to B)\to A]^3 \qquad \dfrac{\dfrac{\dfrac{[A]^1 \quad [\cancel{A}]^2}{\#}\,\uparrow}{B}\,\downarrow}{A\to B}\,{\to}I^1}{A}\,{\to}E \qquad [\cancel{A}]^2}{\#}\,\uparrow}{A}\,\downarrow^2}{((A\to B)\to A)\to A}\,{\to}I^3
$$

Contexts contain positive & negative information. Judgements are positive or dead ends.

A (MILDLY) BILATERAL PROOF OF PEIRCE'S LAW

(in M. Parigot's $\lambda\mu$ calculus)

$$\dfrac{x\,[A]^1 \qquad \alpha\,[\cancel{A}]^2}{\alpha[x]\ \#}\ \uparrow$$

$$\dfrac{\alpha[x]\ \#}{\mu\beta.\alpha[x]\ B}\ \downarrow$$

$$\dfrac{\mu\beta.\alpha[x]\ B}{\lambda x.\mu\beta\alpha[x]\ A \to B}\ \to I^1$$

$$\dfrac{y\,[(A \to B) \to A]^3 \qquad \lambda x.\mu\beta\alpha[x]\ A \to B}{y(\lambda x.\mu\beta.\alpha[x])\ A}\ \to E$$

$$\dfrac{y(\lambda x.\mu\beta.\alpha[x])\ A \qquad \alpha\,[\cancel{A}]^2}{\alpha[y(\lambda x.\mu\beta.\alpha[x])]\ \#}\ \uparrow$$

$$\dfrac{\alpha[y(\lambda x.\mu\beta.\alpha[x])]\ \#}{\mu\alpha.\alpha[y(\lambda x.\mu\beta.\alpha[x])]\ A}\ \downarrow^2$$

$$\dfrac{\mu\alpha.\alpha[y(\lambda x.\mu\beta.\alpha[x])]\ A}{\lambda y.\mu\alpha.\alpha[y(\lambda x.\mu\beta.\alpha[x])]\ ((A \to B) \to A) \to A}\ \to I^3$$

Contexts contain positive & negative information. Judgements are positive or dead ends.

A SYMMETRIC BICATERAL CALCULUS $\bar{\lambda}\mu\tilde{\mu}$ (CURIEN & HERBELIN)

At the typing level, we obtain $LK_{\mu\tilde{\mu}}$ whose typing judgements are:

$$c : (\Gamma \vdash \Delta) \quad \text{— COMMANDS}$$
$$\Gamma \vdash v : A \,|\, \Delta \quad \text{— TERMS}$$
$$\Gamma \,|\, e : A \vdash \Delta \quad \text{— CONTEXTS}$$

CLASHES
ASSERTIONS
DENIALS

and whose typing rules are:

$$\frac{\Gamma \vdash v : A \,|\, \Delta \qquad \Gamma \,|\, e : A \vdash \Delta}{\langle v | e \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{}{\Gamma \,|\, \alpha : A \vdash \alpha : A, \Delta}$$

$$\frac{}{\Gamma, x : A \vdash x : A \,|\, \Delta}$$

$$\frac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma \vdash \mu\beta.c : B \,|\, \Delta}$$

$$\frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}x.c : A \vdash \Delta}$$

$$\frac{\Gamma \vdash v : A \,|\, \Delta \qquad \Gamma \,|\, e : B \vdash \Delta}{\Gamma \,|\, v \cdot e : A \to B \vdash \Delta}$$

$$\frac{\Gamma, x : A \vdash v : B \,|\, \Delta}{\Gamma \vdash \lambda x.v : A \to B \,|\, \Delta}$$

complex contexts / denials.

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FREGE'S BEGRIFFSCHRIFT

A — content, a thought, a proposition.

$\vdash A$ — the assertion that A

If A then B. — this can be asserted, but the A & B are propositions inside the conditional, but are not asserted.

Assertion is a speech act — there are others.

$?A$ — polar question

$?_x A(x)$ — find an x where $A(x)$ question

QUESTION

$!A$ — see to it that A is true

IMPERATIVE

$\S_\beta A$ — β promises to see to it that A

COMMITMENT

CONDITIONAL SPEECH ACTS

If A then is it the case that B?

If A then I promise to B.

If A then please do B.

Are these questions, promises & imperatives? Certainly if the antecedent holds... maybe only then.

If A is a restrictor of more than propositions.

Traditional formal grammars do not respect conditional speech acts — the grammar is independent of the semantics.

$$\frac{A\ \text{prop} \qquad \begin{array}{c}[A\ \text{true}]\\ \vdots \\ B\ \text{prop}\end{array}}{A \supset B\ \text{prop}} \qquad\qquad \frac{A\ \text{prop} \qquad \begin{array}{c}[A\ \text{true}]\\ \vdots \\ B\ \text{promise}\end{array}}{A \supset B\ \text{promise}}$$

These are also entangled, but the dependence is in the other direction!

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TYPE THEORY CAN BE APPLIED IN DIFFERENT WAYS

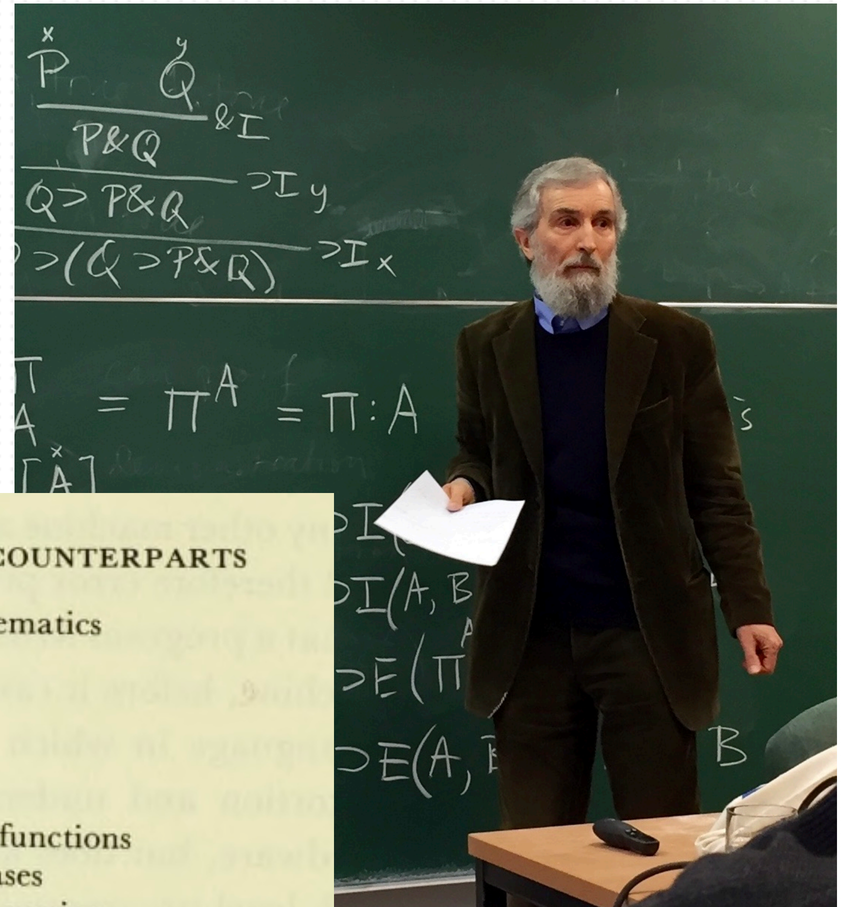

TABLE 1. KEY NOTIONS OF PROGRAMMING WITH MATHEMATICAL COUNTERPARTS

programming	mathematics
program, procedure, algorithm	function
input	argument
output, result	value
$x := e$	$x = e$
$S_1; S_2$	composition of functions
if B **then** S_1 **else** S_2	definition by cases
while B **do** S	definition by recursion
data structure	element, object
data type	set, type
value of a data type	element of a set, object of a type
$a : A$	$a \in A$
integer	Z
real	R
Boolean	$\{0, 1\}$
$(c_1, \ldots, c_n)$	$\{c_1, \ldots, c_n\}$
array [I] **of** T	T^I, $I \to T$
record $s_1 : T_1$; $s_2 : T_2$ **end**	$T_1 \times T_2$
record case $s : (c_1, c_2)$ **of** $c_1 : (s_1 : T_1)$; $c_2 : (S_2 : T_2)$ **end**	$T_1 + T_2$
set of T	$\{0, 1\}^T$, $T \to \{0, 1\}$

PML – Constructive Mathematics & Computer programming (1984)

Computational type theory

sequents classify computational processes

Formal type theory

pure, logic, backed only by the rules

$$\Gamma \vdash t : A,$$

. . . . .

Conceptual type theory

sequents classify cognitive constructions

CONSTRUCTIVE LOGIC

INTUITIONISM — theories of judgement

PML, Dag Prawitz, Göran Sundholm, ...

Dialogical type theory

sequents classify practices & processes of reasoning & justification

NORMATIVE PRAGMATICS

Robert Brandom, Jaroslav Peregrin, ...

HYBRID TYPE THEORY?

Computational type theory
sequents classify computational processes

Formal type theory

$$\Gamma \vdash t : A$$

. . . .

What about applications that encompass these domains? Justifications that include computation, computer aided reasoning, natural language program specification

Conceptual type theory
sequents classify cognitive constructions

Dialogical type theory
sequents classify practices & processes of reasoning & justification

It seems to me that many of these intersections could be fruitful in the years ahead.

Questions?