Cohomology in Synthetic Stone Duality

Hugo Moeneclaey

j.w.w. Felix Cherubini, Thierry Coquand and Freek Geerligs

TYPES 2025
Glasgow

Overview

We work in Synthetic Stone Duality (SSD), using pen and paper.

Overview

We work in Synthetic Stone Duality (SSD), using pen and paper.

$$\text{SSD} = \text{HoTT} + 4 \text{ axioms}.$$

Overview

We work in Synthetic Stone Duality (SSD), using pen and paper.

$$\text{SSD} = \text{HoTT} + 4 \text{ axioms}.$$

Cohomology in HoTT

Given $n : \mathbb{N},\ X : \text{Type},\ A : X \to \text{Ab}$, we define a group $H^n(X, A)$.

$H^n(X, A)$ is the n-th cohomology group of X with coefficient A.

Overview

We work in Synthetic Stone Duality (SSD), using pen and paper.

$$\mathsf{SSD} = \mathsf{HoTT} + 4 \text{ axioms.}$$

Cohomology in HoTT

Given $n : \mathbb{N}, \; X : \mathsf{Type}, \; A : X \to \mathsf{Ab}$, we define a group $H^n(X, A)$.

$H^n(X, A)$ is the *n-th cohomology group of X with coefficient A*.

Our previous work [CCGM24]

- Showed SSD is suitable for synthetic topological study of Stone and compact Hausdorff spaces.
- Proved $H^1(X, \mathbb{Z})$ is well-behaved for $X : \mathsf{CHaus}$.

Overview

Today

$H^n(X, A)$ is well-behaved for X : CHaus and $A : X \to \mathrm{Ab}_{cp}$.

Overview

Today

$H^n(X, A)$ is well-behaved for X : CHaus and $A : X \to \mathsf{Ab}_{cp}$.

Plan

1. Introduce SSD, Stone spaces and compact Hausdorff spaces.

Overview

Today

$H^n(X, A)$ is well-behaved for X : CHaus and $A : X \to \mathrm{Ab}_{cp}$.

Plan

1. Introduce SSD, Stone spaces and compact Hausdorff spaces.
2. Introduce the cohomology groups $H^n(X, A)$.

Overview

Today

$H^n(X, A)$ is well-behaved for X : CHaus and $A : X \to \mathrm{Ab}_{cp}$.

Plan

1. Introduce SSD, Stone spaces and compact Hausdorff spaces.
2. Introduce the cohomology groups $H^n(X, A)$.
3. Introduce overtly discrete types and Barton-Commelin axioms:

$$\Pi_{x:X} I(x) \text{ is well-behaved for } X : \mathrm{CHaus} \text{ and } I : X \to \mathrm{ODisc}.$$

Overview

Today

$H^n(X, A)$ is well-behaved for X : CHaus and $A : X \to \mathrm{Ab}_{cp}$.

Plan

1. Introduce SSD, Stone spaces and compact Hausdorff spaces.
2. Introduce the cohomology groups $H^n(X, A)$.
3. Introduce overtly discrete types and Barton-Commelin axioms:
   $\Pi_{x:X} I(x)$ is well-behaved for X : CHaus and $I : X \to \mathrm{ODisc}$.
4. Explain our main results:
   $H^n(X, A)$ is well-behaved for X : CHaus and $A : X \to \mathrm{Ab}_{\mathrm{ODisc}}$.

An abelian group is overtly discrete iff it is countably presented.

Outline

SSD, Stone spaces and compact Hausdorff spaces

Intoduction to cohomology in HoTT

Overtly discrete types and Barton-Commelin axioms

Cohomology of Stone and compact Hausdorff spaces

Stone spaces

Definition

A type X is a Stone space if it is a sequential limit of finite types.

Stone spaces

Definition

A type X is a Stone space if it is a sequential limit of finite types.

Example 1: Cantor space

The type $2^{\mathbb{N}}$ is a Stone space.

Indeed $2^{\mathbb{N}} = \lim_{i:\mathbb{N}} 2^i$.

Stone spaces

Definition

A type X is a Stone space if it is a sequential limit of finite types.

Example 1: Cantor space

The type $2^{\mathbb{N}}$ is a Stone space.

Indeed $2^{\mathbb{N}} = \lim_{i:\mathbb{N}} 2^i$.

Example 2: Compactification of $\mathbb{N}$

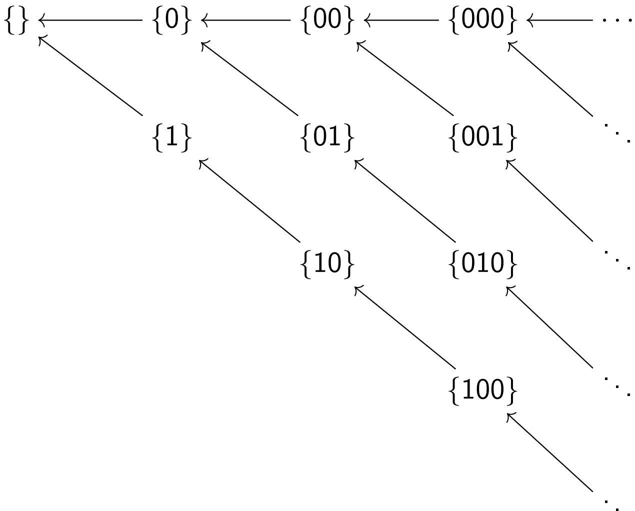
The type:

$$\mathbb{N}_\infty = \{\alpha : 2^{\mathbb{N}} \mid \alpha \text{ hits 1 at most once}\}$$

is a Stone space.

Indeed $\mathbb{N}_\infty$ is the limit of:

$$\mathsf{Fin}(1) \xleftarrow{-1} \mathsf{Fin}(2) \xleftarrow{-1} \mathsf{Fin}(3) \xleftarrow{-1} \mathsf{Fin}(4) \xleftarrow{-1} \cdots$$



Synthetic Stone duality

Axiom 1a: Scott continuity

If $(S_k)_{k:\mathbb{N}}$ is a tower of finite types, then any map in $(\lim_k S_k) \to 2$ merely factors through an S_k.

Synthetic Stone duality

Axiom 1a: Scott continuity

If $(S_k)_{k:\mathbb{N}}$ is a tower of finite types, then any map in $(\lim_k S_k) \to 2$ merely factors through an S_k.

Axiom 1b: Markov's principle

If $(D_k)_{k:\mathbb{N}}$ are decidable propositions, $\neg(\forall_{k:\mathbb{N}} D_k) \to \exists(k : \mathbb{N}). \neg D_k$.

Synthetic Stone duality

Axiom 1a: Scott continuity

If $(S_k)_{k:\mathbb{N}}$ is a tower of finite types, then any map in $(\lim_k S_k) \to 2$ merely factors through an S_k.

Axiom 1b: Markov's principle

If $(D_k)_{k:\mathbb{N}}$ are decidable propositions, $\neg(\forall_{k:\mathbb{N}} D_k) \to \exists(k : \mathbb{N}). \neg D_k$.

Axiom 2: Weak König's lemma

If $(S_k)_{k:\mathbb{N}}$ is a tower of inhabited finite types, then $\|\lim_k S_k\|$.

Synthetic Stone duality

Axiom 1a: Scott continuity

If $(S_k)_{k:\mathbb{N}}$ is a tower of finite types, then any map in $(\lim_k S_k) \to 2$ merely factors through an S_k.

Axiom 1b: Markov's principle

If $(D_k)_{k:\mathbb{N}}$ are decidable propositions, $\neg(\forall_{k:\mathbb{N}} D_k) \to \exists(k : \mathbb{N}).\neg D_k$.

Axiom 2: Weak König's lemma

If $(S_k)_{k:\mathbb{N}}$ is a tower of inhabited finite types, then $\|\lim_k S_k\|$.

Axiom 4: Dependent choice

Synthetic Stone duality

Axiom 1a: Scott continuity

If $(S_k)_{k:\mathbb{N}}$ is a tower of finite types, then any map in $(\lim_k S_k) \to 2$ merely factors through an S_k.

Axiom 1b: Markov's principle

If $(D_k)_{k:\mathbb{N}}$ are decidable propositions, $\neg(\forall_{k:\mathbb{N}} D_k) \to \exists(k : \mathbb{N}). \neg D_k$.

Axiom 2: Weak König's lemma

If $(S_k)_{k:\mathbb{N}}$ is a tower of inhabited finite types, then $\|\lim_k S_k\|$.

Axiom 4: Dependent choice

Axiom 3: Local choice

Assume given S : Stone and $Y : S \to \text{Type}$ such that $\Pi_{s:S} \| Y(s) \|$.
Then there exists T : Stone and $p : T \twoheadrightarrow S$ such that $\Pi_{t:T} Y(p(t))$.

Compact Hausdorff spaces

Stone spaces are not stable under quotients.

Definition

A set X is a compact Hausdorff space if:

- Its identity types are Stone spaces.
- There exists S : Stone and $S \twoheadrightarrow X$.

Compact Hausdorff spaces

Stone spaces are not stable under quotients.

Definition

A set X is a compact Hausdorff space if:

- Its identity types are Stone spaces.
- There exists S : Stone and $S \twoheadrightarrow X$.

Example: The unit interval

The type $\mathbb{I} = [0, 1]$ is a compact Hausdorff space.

Indeed $\mathbb{I}$ is a quotient of $2^{\mathbb{N}}$.

Outline

SSD, Stone spaces and compact Hausdorff spaces

Intoduction to cohomology in HoTT

Overtly discrete types and Barton-Commelin axioms

Cohomology of Stone and compact Hausdorff spaces

Delooping abelian groups

Fix A an abelian group. We define $K(A, 0) = A$.

Proposition

Given $n > 0$, there is a unique pointed type $K(A, n)$ such that:

- $K(A, n)$ is $(n\text{-}1)$-connected and n-truncated.
- $\Omega^n K(A, n) = A$.

$K(A, n)$ is called the n-th delooping of A.

Cohomology groups

Definition: Cohomology

Given $n : \mathbb{N}$, X : Type and $A : X \to \mathrm{Ab}$, we define

$$H^n(X, A) = \|\Pi_{x:X}\, K(A_x, n)\|_0.$$

Cohomology groups

Definition: Cohomology

Given $n : \mathbb{N}$, X : Type and $A : X \to \mathrm{Ab}$, we define

$$H^n(X, A) = \| \Pi_{x:X} K(A_x, n) \|_0.$$

Remark: Why cohomology?

- If $H^n(X, A) = 0$ then we can use some choice on X.
- There exists many tools to compute cohomology.

Outline

Overtly discrete types

We want A such that $H^n(X, A)$ is well-behaved for X : CHaus.

Overtly discrete types

We want A such that $H^n(X, A)$ is well-behaved for X : CHaus.

Idea

We assume A takes value in overtly discrete abelian groups.

Overtly discrete types

We want A such that $H^n(X, A)$ is well-behaved for X : CHaus.

Idea

We assume A takes value in overtly discrete abelian groups.

Definition

A type is overtly discrete if it is a sequential colimit of finite types.

An abelian group is overtly discrete iff it is countably presented.

Tychonov and its dual

We prove Barton-Commelin's condensed type theory axioms.

Tychonov and its dual

We prove Barton-Commelin's condensed type theory axioms.

Lemma: Tychonov

If I : ODisc and $X : I \to$ CHaus, then $\Pi_{i:I} X_i$ is compact Hausdorff.

Tychonov and its dual

We prove Barton-Commelin's condensed type theory axioms.

Lemma: Tychonov

If I : ODisc and $X : I \to$ CHaus, then $\Pi_{i:I} X_i$ is compact Hausdorff.

Proposition: Tychonov's dual

If X : CHaus and $I : X \to$ ODisc, then $\Pi_{x:X} I_x$ is overtly discrete.

This is encouraging. We have better!

Scott continuity

Definition

We have a category $\mathcal{C}$ where:

$$\begin{aligned} \mathrm{Ob}_{\mathcal{C}} &= \Sigma(X : \mathsf{CHaus}).\, X \to \mathsf{ODisc} \\ \mathrm{Hom}_{\mathcal{C}}((X, I), (Y, J)) &= \Sigma(f : Y \to X).\, \Pi_{y:Y}\, I_{f(x)} \to J_x \end{aligned}$$

Scott continuity

Definition

We have a category $\mathcal{C}$ where:

$$\begin{aligned} \mathrm{Ob}_{\mathcal{C}} &= \Sigma(X : \mathsf{CHaus}).\, X \to \mathsf{ODisc} \\ \mathrm{Hom}_{\mathcal{C}}((X, I), (Y, J)) &= \Sigma(f : Y \to X).\, \Pi_{y:Y}\, I_{f(x)} \to J_x \end{aligned}$$

Theorem: Generalized Scott continuity

The functor $\Pi : \mathcal{C} \to \mathsf{ODisc}$ commutes with sequential colimits.

Outline

Čech cohomology

Definition

A Čech cover consists of X : CHaus and S : Stone with $p : S \twoheadrightarrow X$.

Definition

Given a Čech cover $p : S \twoheadrightarrow X$ and $A : X \to \mathrm{Ab}_{cp}$, we define $\check{H}^n(X, S, A)$ as the n-th cohomology group of

$$\Pi_{x:X} A_x^{S_x} \to \Pi_{x:X} A_x^{S_x^2} \to \Pi_{x:X} A_x^{S_x^3} \to \cdots .$$

$\check{H}^n(X, S, A)$ is called the *n*-th Čech cohomology group of X with coefficient in A.

Main results

Theorem: Cohomology vanishing for Stone spaces

Given $n > 0$, S : Stone and $A : S \to \mathrm{Ab}_{cp}$, we have that

$$H^n(S, A) = 0.$$

Theorem: Čech and regular cohomology agree on CHaus

Given a Čech cover $p : S \twoheadrightarrow X$ and $A : X \to \mathrm{Ab}_{cp}$, we have that

$$H^n(X, A) = \check{H}^n(X, S, A).$$

Applications

Lemma: Cohomology of the interval

For $A : \mathrm{Ab}_{cp}$, we have that

$$H^n(\mathbb{I}, A) = \begin{cases} A & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma: Cohomology of the spheres

For $\mathbb{S}^k = \{x_0, \ldots, x_k : \mathbb{R} \mid \Sigma_i\, x_i^2 = 1\}$ and $A : \mathrm{Ab}_{cp}$, we have that

$$H^n(\mathbb{S}^k, A) = \begin{cases} A & \text{if } n = 0 \text{ or } n = k \\ 0 & \text{otherwise.} \end{cases}$$

This extends to all countable topological CW complex.

Axiom 1a: Scott continuity

If $(S_k)_{k:\mathbb{N}}$ is a tower of finite types, then any map in $(\lim_k S_k) \to 2$ merely factors through an S_k.

Axiom 1b: Markov's principle

If $(D_k)_{k:\mathbb{N}}$ are decidable propositions, $\neg(\forall_{k:\mathbb{N}} D_k) \to \exists(k : \mathbb{N}). \neg D_k$.

Axiom 2: Weak König's lemma

If $(S_k)_{k:\mathbb{N}}$ is a tower of inhabited finite types, then $\|\lim_k S_k\|$.

Axiom 4: Dependent choice

If $(X_k)_{k:\mathbb{N}}$ is a tower with surjective maps, then $\lim_k X_k \twoheadrightarrow X_0$.

Axiom 3: Local choice

Assume given S : Stone and $Y : S \to \mathrm{Type}$ such that $\Pi_{s:S}\|Y(s)\|$.
Then there exists T : Stone and $p : T \twoheadrightarrow S$ such that $\Pi_{t:T} Y(p(t))$.