Arrow algebras
(jww Marcus Briët and Umberto Tarantino)

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Section 1

Locales

Locales (point-free spaces)

Locales

A poset $(L, \preccurlyeq)$ is a *frame* or *locale*, if thas finite meets and arbitrary joins with the infinite joins distributing over the finite meets:

$$x \curlywedge \bigcurlyvee_i y_i = \bigcurlyvee_i (x \curlywedge y_i).$$

A *morphism of frames* $(L, \preccurlyeq) \to (M, \preccurlyeq)$ is a monotone function $f : L \to M$ which preserves finite meets and arbitrary joins, thus yielding a category Frm. The category Loc of locales is defined as Frm^{op}.

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There is a functor

$$\Omega : \mathsf{Top} \to \mathsf{Loc}$$

which sends a topological space to its poset of opens. This functor has a right adjoint and this adjunction restricts to an equivalence between *sober spaces* and *locales with enough points*.

Triposes

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Tripos (Hyland, Johnstone, Pitts)

Write PreHey for the category of *preHeyting algebras*. A *tripos* is a pseudofunctor $P : \mathrm{Sets} \to \mathrm{PreHey}^{\mathrm{op}}$ such that:

- for each function $f : Y \to X$, the operation $Pf : PX \to PY$ has both adjoints satisfying the Beck-Chevally condition.
- There is a set Prop and an element $\top \in P(\mathrm{Prop})$ such that for any $A \in P(X)$ there is some map $a : X \to \mathrm{Prop}$ such that $P(a)(\top) \cong A$.

Think: model of higher-order intuitionistic logic with an impredicative and intensional Prop.

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Example: localic tripos

From any locale $(L, \preccurlyeq)$ we obtain a tripos P_L with $P_L(X) := X \to L$ with the pointwise ordering.

Beyond locales?

Tripos-to-topos construction

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There are many interesting of non-localic triposes.

Effective tripos

For any set X, define $P_E(X)$ as the set of functions $X \to \mathrm{Pow}(\mathbb{N})$. If $\varphi, \psi : X \to \mathrm{Pow}(\mathbb{N})$ are two such functions, we will write $\varphi \preccurlyeq \psi$ if there is a partial recursive function f such that for any $x \in X$ and $n \in \varphi(x)$ we have that $f(n)$ is defined and belongs to $\psi(x)$.

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Motivating question

Can we generalise the theory of locales in such a way that other toposes like the effective topos can also be understood as "sheaves over a generalised locale"?

Section 2

Arrow algebras

Arrow structure

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An *arrow structure* is a complete poset $(A, \preccurlyeq)$ together with a binary operation $\to : A \times A \to A$ satisfying the following condition:

If $a' \preccurlyeq a$ and $b \preccurlyeq b'$ then $a \to b \preccurlyeq a' \to b'$.

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Examples

- Every locale is a complete Heyting algebra with implication given by:

$$x \rightarrow y := \bigvee \{z : x \curlywedge z \preccurlyeq y\}.$$

- We also have $(\mathrm{Pow}(\mathbb{N}), \subseteq)$ with

$$X \rightarrow Y = \{e : (\forall x \in X)\, e \cdot x \downarrow \text{ and } e \cdot x \in Y\}.$$

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Intuition

We think of the elements of A as truth values or bits of evidence, and we refer to $\preccurlyeq$ as the "evidential ordering" ("subtyping ordering").

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Separators

Let $A = (A, \preccurlyeq, \to)$ be an arrow structure. A *separator* on A is a subset $S \subseteq A$ such that the following are satisfied:

(1) If $a \in S$ and $a \preccurlyeq b$, then $b \in S$.

(2) If $a \to b \in S$ and $a \in S$, then $b \in S$.

(3) S contains the combinators ("tautologies") k, s and a.

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Here k, s and a are defined as follows:

$$\begin{aligned}
\mathrm{k} &:= \bigcurlywedge_{a,b} a \rightarrow b \rightarrow a \\
\mathrm{s} &:= \bigcurlywedge_{a,b,c} (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow c) \\
\mathrm{a} &:= \bigcurlywedge_{a \in A, B \subseteq \mathrm{Im}(\rightarrow)} \Big(\bigcurlywedge_{b \in B} a \rightarrow b\Big) \rightarrow a \rightarrow \bigcurlywedge_{b \in B} b.
\end{aligned}$$

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Examples

1. A frame $(L, \preccurlyeq)$ with $S = \{\top\}$.
2. A frame $(L, \preccurlyeq)$ with S an arbitrary filter.
3. The effective arrow algebra $(\mathrm{Pow}(\mathbb{N}), \subseteq, \to, \mathrm{Pow}_i(\mathbb{N}))$.

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We will now explain how any arrow algebra gives rise to a tripos (and hence a topos).

Arrow tripos

Proposition

Let $A = (A, \preccurlyeq, \rightarrow, S)$ be an arrow algebra. If we preorder A as follows:

$$a \vdash b :\Longleftrightarrow a \rightarrow b \in S,$$

then A carries the structure of a preHeyting algebra.

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In this preHeyting algebra the implication is given by $\to$. We think of $\vdash$ as giving us the *logical ordering*.

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If $A = (A, \preccurlyeq, \to, S)$ is an arrow algebra and X is a set, then A^X is an arrow algebra as well: implication and the order can be defined pointwise, while

$$\varphi : X \to A \in S^X :\Longleftrightarrow \bigcurlywedge_{x \in X} \varphi(x) \in S.$$

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If we put $PX = (A^X, \vdash_{S^X})$, then this defines a tripos: we write $\mathrm{AT}(A)$ for the *arrow tripos* associated to A. This recovers both localic triposes as well as the effective tripos.

Section 3

More examples: pcas

Pcas

Partial combinatory algebra (pca)

A *partial combinatory algebra (pca)* is a quadruple $(A, \cdot, \leq, A^{\#})$ where:

- $(A, \leq)$ is a poset.
- $\cdot$ is a partial binary operation, such that if $a'b'$ is defined and $a \leq a'$ and $b \leq b'$, then ab is also defined and $ab \leq a'b'$.
- $A^{\#}$ is a subset $A^{\#} \subseteq A$ such that for all $a, b, c \in A$:
  - (i) if $a, b \in A^{\#}$ and ab is defined, then $ab \in A^{\#}$.
  - (ii) if $a \leq b$ and $a \in A^{\#}$, then $b \in A^{\#}$.
  - (iii) there are elements $\mathsf{k}, \mathsf{s} \in A^{\#}$ satisfying:
    - (1) $\mathsf{k}ab\downarrow$ and $\mathsf{k}ab \leq a$;
    - (2) $\mathsf{s}ab\downarrow$;
    - (3) if $ac(bc)\downarrow$, then $\mathsf{s}abc\downarrow$ and $\mathsf{s}abc \leq ac(bc)$.

Remark

The usual notion of a pca is more restrictive. For our purposes, the definition above, which is also the one used in Jetze Zoethout's PhD thesis, is quite convenient.

Tripos from a pca

Examples

1. K_1: the set of natural numbers with Kleene application ($n \cdot m$ is the outcome of the n-th Turing machine on input m, whenever defined) and the discrete order. All elements belong to the filter.
2. Terms in the untyped λ-calculus and $M \leq N$ if $M \twoheadrightarrow_\beta N$. All elements belong to the filter.
3. Write $\mathbb{P} = \mathcal{P}(\mathbb{N})$ and fix a computable bijection $[-] : \mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathbb{N} \to \mathbb{N}$. Then $X \cdot Y = \{z : (\exists \gamma \in \mathcal{P}_{\text{fin}}(Y))\, [\gamma, z] \in X\}$ defines a total binary application on $\mathbb{P}$ and $\mathbb{P}^\# = \{X \in \mathbb{P} : X \text{ is recursively enumerable}\}$ defines a filter.

If $\mathbb{P}$ is a pca, then $(D\mathbb{P}, \subseteq, \to, S)$ is an arrow algebra, where:

- $D\mathbb{P}$ is the collection of downsets in P,
- $X \to Y := \{z \in P : (\forall x \in X)\, zx\downarrow \text{ and } zx \in Y\}$,
- $S = \{X \in D\mathbb{P} : (\exists x \in X)\, x \in F\}$.

Section 4

Nuclei and morphisms

Nuclei

Nucleus

Let $A = (A, \preccurlyeq, \to, S)$ be an arrow algebra. A mapping $j : A \to A$ will be called a *nucleus* if the following three properties are satisfied:

(1) $a \preccurlyeq b$ implies $ja \preccurlyeq jb$ for all $a, b \in A$.

(2) $\curlywedge_{a \in A} a \to ja \in S$.

(3) $\curlywedge_{a,b \in A} (a \to jb) \to (ja \to jb) \in S$.

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Examples

Let $A = (A, \preccurlyeq, \to, S)$ be an arrow algebra and $a \in A$. Then the following define nuclei:

- $jx = (x \to a) \to a$
- $jx = a \to x$
- $jx = x + a$, where $+$ is the join in the logical ordering.
- But also: Lifschitz and covering modalities.

Subalgebras from nuclei

Proposition

Let $(A, \preccurlyeq, \rightarrow, S)$ be an arrow algebra and $j : A \rightarrow A$ be a nucleus on it. Then $A_j = (A, \preccurlyeq, \rightarrow_j, S_j)$ with

$$\begin{aligned} a \rightarrow_j b &\quad :\equiv \quad a \rightarrow jb \\ a \in S_j &\quad :\Leftrightarrow \quad ja \in S \end{aligned}$$

is also an arrow algebra.

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Theorem

The arrow tripos associated to A_j is a subtripos of the one associated to A. Indeed, any subtripos of $\mathrm{AT}(A)$ is of this form.

Morphisms of arrow algebras

Tarantino has developed a notion of *morphisms of arrow algebras*. It has the following properties:

- Morphisms of arrow algebras between locales coincide with locale morphisms.
- Morphisms between arrow algebras deriving from pcas correspond to computationally dense morphisms of pcas.
- Morphisms of arrow algebras correspond to geometric morphisms between the associated triposes.
- Morphisms between arrow algebras can be factored as a surjection followed by an embedding, where these surjections and embeddings induce surjections and embeddings on the level of triposes. The embeddings of arrow algebras are induced by (unique) nuclei.

Please check out his paper!

Section 5

Comparison to work of Miquel

Implicative algebras

Our work on arrow algebras is heavily inspired by the work of Alexandre Miquel on *implicative algebras*.

Implicative algebra (Miquel)

An *implicative algebra* is an arrow algebra satisfying

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2. Every arrow algebra is equivalent to an implicative algebra.
3. However, there are many naturaly occurring examples of arrow algebras which are not implicative algebras. For instance, if $\mathbb{P}$ is a pca, then $D\mathbb{P}$ is an implicative algebra iff the application in $\mathbb{P}$ is total.

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2. Every arrow algebra is equivalent to an implicative algebra.
3. However, there are many naturaly occurring examples of arrow algebras which are not implicative algebras. For instance, if $\mathbb{P}$ is a pca, then $D\mathbb{P}$ is an implicative algebra iff the application in $\mathbb{P}$ is total.
4. Also, we have a notion of morphism of arrow algebras and a neat factorisation of these morphisms into surjections and inclusions.

THANK YOU!

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