Categorical Normalization by Evaluation: A Novel Universal Property of Syntax

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Outline

1. Motivation
2. Universal Property of Quotiented Syntax
3. Sound Normalization
4. Universal Property of Unquotiented Syntax relative to Quotiented Syntax
5. Strongly Complete Normalization
6. Gluing
7. Correctness
8. Rocq Formalization
9. Future Work

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  - Fiore: Fully categorical, *ad hoc* analysis of normal forms, no universal property, full correctness
- Our approach: Fully categorical, no *ad hoc* analysis of normal forms, universal property(/ies), full correctness

Motivation

We present a slick, new, (ᴘ-)categorical construction of NbE with full correctness.

Free CCC over Single Basetype

Free CCC

$$\mathcal{F}_0 \triangleq \text{Contexts}$$
$$\mathcal{F}_1(\Gamma, \Delta) \triangleq \text{Substitutions} : \Gamma \vdash \Delta \text{ “/” } \cong_{\beta\eta}$$

$$\mathcal{F} \xrightarrow[\forall I]{} \mathbb{M}$$

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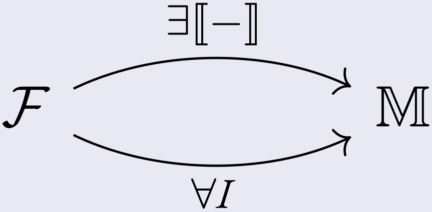

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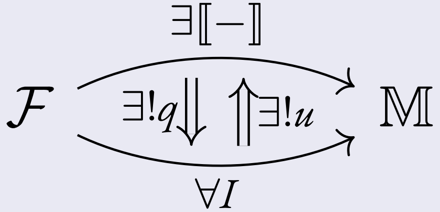

Abstract Normalization

Normalization Function

$$\mathsf{nf}_I(\Gamma \vdash \sigma : \Delta) \triangleq q_\Delta \circ [\![\sigma]\!] \circ u_\Gamma : I(\Gamma) \to I(\Delta)$$

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Soundness

$$\begin{aligned}
\mathsf{nf}_I(\Gamma \vdash \sigma : \Delta) &\equiv q_\Delta \circ [\![\sigma]\!] \circ u_\Gamma \\
&\sim_{\mathbb{M}} q_\Delta \circ u_\Delta \circ I(\sigma) \\
&\sim_{\mathbb{M}} \mathrm{id}_\Delta \circ I(\sigma) \\
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Weak Completeness

$$\Gamma \vdash \sigma \cong_{\beta\eta} \sigma' : \Delta \Rightarrow \mathsf{nf}_I(\sigma) \sim_{\mathbb{M}} \mathsf{nf}_I(\sigma')$$

Concrete Normalization

Choice of $\mathbb{M}$ and I

$\mathbb{M}$	$I : \mathcal{F} \to \mathbb{M}$
~~$\mathcal{F}$~~	~~$\mathrm{Id} : \mathcal{F} \to \mathcal{F}$~~
$\widehat{\mathcal{F}}$	よ $: \mathcal{F} \to \widehat{\mathcal{F}}$

Cartesian Pre-Closure

Cartesian Pre-Closed Category

A category, $\mathbb{C}$, is Cartesian-pre-closed when:

- it is Cartesian;
- it has a pre-exponential operator on objects:
$$(-) \Rightarrow (=) : \mathbb{C}_0 \times \mathbb{C}_0 \to \mathbb{C}_0;$$
- such that there are maps natural in c:
$$\mathbb{C}(c \times a, b) \Rightarrow \mathbb{C}(c, a \Rightarrow b)$$
$$\mathbb{C}(c, a \Rightarrow b) \Rightarrow \mathbb{C}(c \times a, b).$$

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Cartesian Pre-Closed Functor

A functor, $F : \mathbb{C} \to \mathbb{D}$, with $\mathbb{C}$ Cartesian-pre-closed and $\mathbb{D}$ Cartesian-closed, is Cartesian-pre-closed when:

- it is Cartesian; and
- there is a family of maps that weakly preserves pre-exponential structure:
$$\tilde{e} : \big(F(a) \Rightarrow F(b)\big) \to F(a \Rightarrow b).$$

Free Cartesian Pre-Closed Category

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$$\mathcal{A}_0 \triangleq \text{Contexts}$$

$$\mathcal{A}_1(\Gamma, \Delta) \triangleq \text{Substitutions} : \Gamma \vdash \Delta \text{ “/” } \equiv_\alpha$$

$$\mathcal{A} \xrightarrow{\forall \bar{I}} \overline{\mathbb{M}}$$

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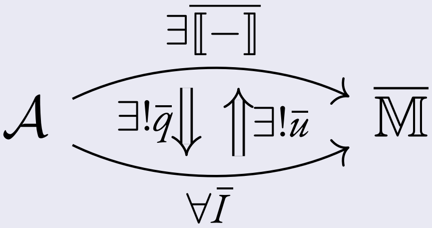

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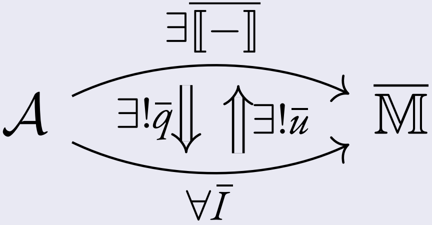

Renamings (Free Cartesian Category over $\mathbb{T}$)

$$\mathcal{R}_0 \triangleq \text{Contexts}$$

$$\mathcal{R}_1(\Gamma, \Delta) \triangleq \text{Renamings} : \Gamma \vdash \Delta$$

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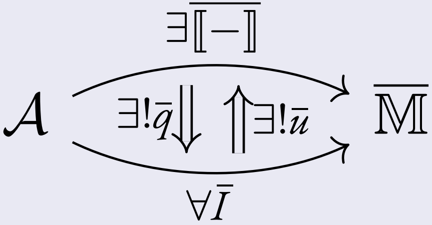

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Inclusion and Quotient Functors

$$i : \mathcal{R} \to \mathcal{A}$$

$$j : \mathcal{A} \to \mathcal{F}$$

Concrete Normalization

Choice of $\mathbb{M}$ and I

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$\widehat{\mathcal{A}}$	$\langle j \rangle : \mathcal{F} \to \widehat{\mathcal{A}}$

$$\langle j \rangle(\Delta)(\Gamma) \triangleq \mathcal{F}_1(j^{\mathrm{op}}(\Gamma), \Delta)$$

Stronger Universal Property for $j : \mathcal{A} \to \mathcal{F}$

$\mathcal{R} \triangleq$ Renamings
$\mathcal{A} \triangleq$ Substitutions "/" $\equiv_\alpha$
$\mathcal{F} \triangleq$ Substitutions "/" $\cong_{\beta\eta}$

$$\begin{array}{c} \mathcal{A} \\ {\scriptstyle j}\big\downarrow \\ \mathcal{F} \end{array}$$

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$$\mathcal{A} \xrightarrow{j} \mathcal{F} \qquad\qquad \mathcal{A} \xrightarrow{\forall \tilde{I}} \mathbb{M}$$

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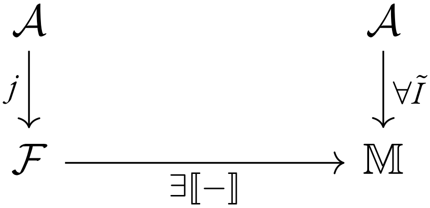

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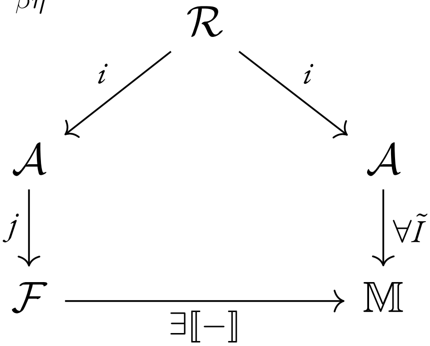

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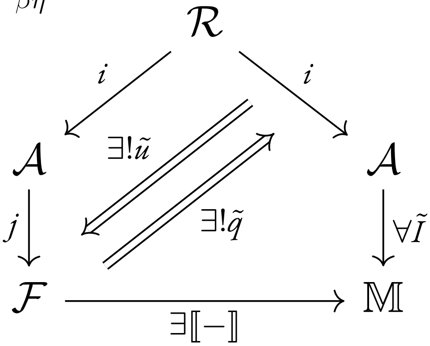

Abstract Normalization 2.0

Normalization Function

$$\widetilde{\mathsf{nf}}_{\tilde{I}}\,(\Gamma \vdash \sigma : \Delta) \triangleq \tilde{q}_\Delta \circ [\![\sigma]\!] \circ \tilde{u}_\Gamma : \tilde{I}\,(\Gamma) \to \tilde{I}\,(\Delta)$$

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Concrete Normalization 2.0

Choice of $\mathbb{M}$ and $\tilde{I}$

$\mathbb{M}$	$\tilde{I} : \mathcal{A} \to \mathbb{M}$
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Strong Completeness

With $\mathbb{M} \triangleq \widehat{\mathcal{A}}$ and $\tilde{I} \triangleq よ$:

$$\Gamma \vdash \sigma \cong_{\beta\eta} \sigma' : \Delta \Rightarrow \widetilde{\mathsf{nf}}_{よ}(\sigma) \equiv_{\alpha} \widetilde{\mathsf{nf}}_{よ}(\sigma')$$

Have soundness but no strong completeness;

OR,

have strong completeness but no soundness.

Gluing

Choice of $\mathbb{M}$ and $\tilde{I}$

$$\mathbb{M} \triangleq \widehat{\mathcal{A}} \downarrow \widehat{\mathcal{A}} \qquad (\cong [\mathcal{A}^{\mathsf{op}}, \mathbf{Set}]^{\to})$$

$$\tilde{I} \triangleq \left\langle よ \downarrow \langle j \rangle j \right\rangle : \mathcal{A} \to \widehat{\mathcal{A}} \downarrow \widehat{\mathcal{A}}$$

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Normalization Functions

$$\widetilde{\mathsf{nf}}_{\mathrm{D}}(\Gamma \vdash \sigma : \Delta) \triangleq \mathrm{Dom}(\tilde{q}_{\Delta} \circ [\![\sigma]\!] \circ \tilde{u}_{\Gamma})_{\Gamma}(\mathrm{id}_{\Gamma}) \quad : \mathcal{A}(\Gamma, \Delta)$$

$$\widetilde{\mathsf{nf}}_{\mathrm{C}}(\Gamma \vdash \sigma : \Delta) \triangleq \mathrm{Cod}(\tilde{q}_{\Delta} \circ [\![\sigma]\!] \circ \tilde{u}_{\Gamma})_{\Gamma}(\mathrm{id}_{\Gamma}) \quad : \mathcal{F}(\Gamma, \Delta)$$

The Normalizer's ~~Dilemma?~~ Triumph!

Normalization Functions

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Correctness

- $\widetilde{\mathsf{nf}}_{\mathrm{D}}$ is strongly complete. $\left[\sigma \cong_{\beta\eta} \sigma' \Rightarrow \widetilde{\mathsf{nf}}_{\mathrm{D}}(\sigma) \equiv_\alpha \widetilde{\mathsf{nf}}_{\mathrm{D}}(\sigma')\right]$
- $\widetilde{\mathsf{nf}}_{\mathrm{C}}$ is sound. $\left[\widetilde{\mathsf{nf}}_{\mathrm{C}}(\sigma) \cong_{\beta\eta} \sigma\right]$
- $\widetilde{\mathsf{nf}}_{\mathrm{D}}$ and $\widetilde{\mathsf{nf}}_{\mathrm{C}}$ agree extensionally. $\left[j(\widetilde{\mathsf{nf}}_{\mathrm{D}}(\sigma)) \cong_{\beta\eta} \widetilde{\mathsf{nf}}_{\mathrm{C}}(\sigma)\right]$
- $\widetilde{\mathsf{nf}}_{\mathrm{D}}$ is sound. $\left[j(\widetilde{\mathsf{nf}}_{\mathrm{D}}(\sigma)) \cong_{\beta\eta} \sigma\right]$

Rocq Formalization

- P-Category Theory used.
- SProp used for PER valuation.
- Provides effective normalization procedure for terms & substitutions.
- De Bruijn indices used for representation of variables.

Future Work

- Analyze naturality of $\tilde{q}$ and $\tilde{u}$ more carefully.
- Connect more closely with traditional gluing techniques (*e.g.*, of Fiore).
- Lift to coproducts.
- Lift to non-simple type theories.

For more information see:

`arXiv:2505.07780`