Internalized Parametricity via Lifting Universals

Work in progress

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Parametricity

Deep principle intricately connected to dependent typing

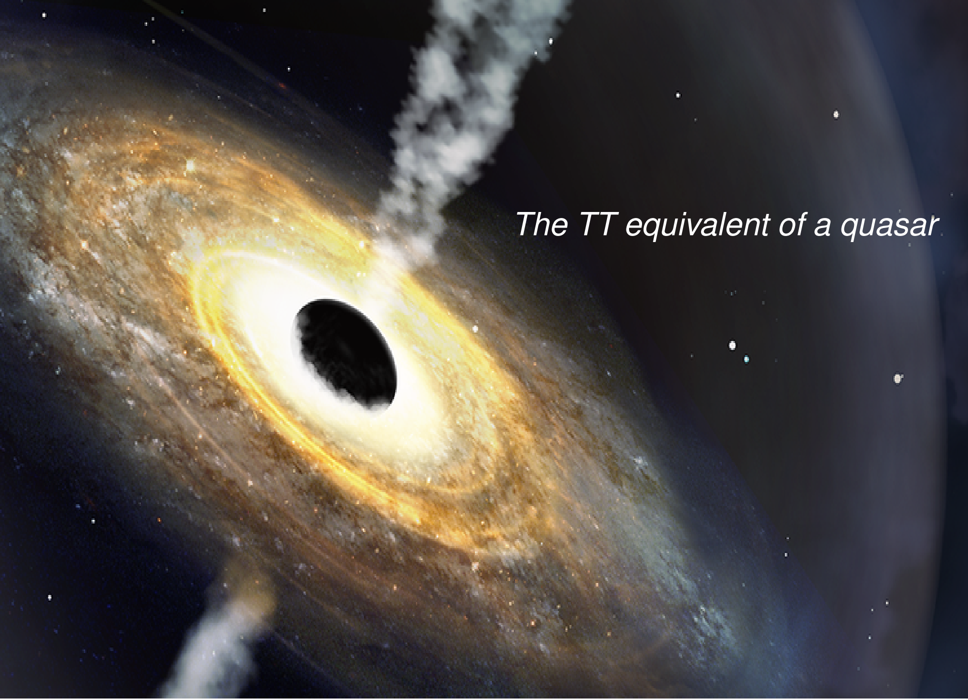
Like a pervasive form of induction across all types

Proposed by Reynolds for parametric polymorphism

“Types, Abstraction, and Parametric Polymorphism” 1983

Essence of Wadler’s “Theorems for Free!” 1989

Ecstatically great axiom...


The TT equivalent of a quasar

Details

1) Given $t : A$, parametricity of arity n says this inhabited:

$$[\![A]\!]^n \underbrace{t \cdots t}_{n}$$

where $[\![A]\!]^n$ is interpretation of A as arity-n logical relation

2) Internal: $[\![A]\!]$ in the theory, not just meta-level operation

What is so cool again?

A pure type system + internal parametricity (π):

- Small core theory
  minimalistic, small trusted computing base
- No fixed notion of inductive datatype
  λ*-encode, derive induction principles from* π
- Theorems for free
  lacking in mainstream current proof assistants

Selected previous work

- ▷ "Realizability and Parametricity in Pure Type Systems"
  Bernardy, Lasson 2011

  Define relational semantics for terms of arbitrary PTS

- ▷ "Computational Interpretation of Parametricity"
  Bernardy, Moulin 2012

  Parametricity internalized as construct $[\![t]\!]$

  Identify technical issue with iterated parametricity $[\![[\![t]\!]]\!]$

  Explicitly treat groups of related variables (hypercubes)

- ▷ "Internal Parametricity, without an Interval"
  Altenkirch, Chamoun, Kaposi, Shulman 2024

  Similar approach, but geometry kept implicit

Motivation for present work

Reap benefits of internal parametricity, with a simpler theory

Theories of Bernardy-Moulin, Altenkirch et al. technically complex

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Why?

Meaning and renamings

- In Bernardy-Lasson, $[\![A]\!]^n$ uses $n+1$ renamings.

  n for arguments to relation, 1 more for proof of relatedness

- For example ($n = 2$):

$$
\begin{array}{rcl}
[\![\Pi\, x : A\,.\,B]\!]_{\rho_0,\rho_1,\mathring{\rho}} & = & \lambda\, f_0\ f_1 : \Pi\, x : A\,.\,B\,. \\
& & \quad \Pi\, x_0 : A_0\,.\,\Pi\, x_1 : A_1\,.\,\Pi\, \mathring{x} : [\![A]\!]_{\rho_0,\rho_1,\mathring{\rho}}\ x_0\ x_1\,. \\
& & \quad [\![B]\!]_{\rho_0[x\mapsto x_0],\rho_1[x\mapsto x_1],\mathring{\rho}[x\mapsto \mathring{x}]}\ (f_0\ x_0)\ (f_1\ x_1)
\end{array}
$$

  where:

$$
\begin{array}{rcl}
A_0 & = & \rho_0 \cdot A \\
A_1 & = & \rho_1 \cdot A
\end{array}
$$

- Renamings $[x \mapsto x_0], [x \mapsto x_1], [x \mapsto \mathring{x}]$

The problem of iterated π

- Let $\mathcal{T} := \Pi\, X : \star\, .\, \Pi\, a : X\, .\, \llbracket X \rrbracket^1\ a$

  Type expressing unary parametricity

- What should $\llbracket \mathcal{T} \rrbracket^2$ (binary relational meaning of $\mathcal{T}$) be?

$$
\begin{aligned}
&\lambda\, f_0\ f_1 : \mathcal{T}\, . \\
&\quad \Pi\, X_0\ X_1 : \star\, .\, \Pi\, \mathring{X} : X_0 \to X_1 \to \star\, . \\
&\quad \Pi\, a_0 : X_0\, .\, \Pi\, a_1 : X_1\, .\, \Pi\, \mathring{a} : \mathring{X}\ a_0\ a_1\, . \\
&\quad \llbracket \llbracket X \rrbracket^1 \rrbracket^2_{\bar{\rho}}\ a_0\ a_1\ \mathring{a}\ (f_0\ X_0\ a_0)\ (f_1\ X_1\ a_1)
\end{aligned}
$$

The problem of iterated π

$$
\begin{array}{lcl}
\mathcal{T} & := & \Pi\, X : \star\,.\, \Pi\, a : X\,.\, \llbracket X \rrbracket^1\ a \\
\llbracket \mathcal{T} \rrbracket^2 & = & \lambda\, f_0\ f_1 : \mathcal{T}\,. \\
 & & \quad \Pi\, X_0\ X_1 : \star\,.\, \Pi\, \mathring{X} : X_0 \to X_1 \to \star\,. \\
 & & \quad \Pi\, a_0 : X_0\,.\, \Pi\, a_1 : X_1\,.\, \Pi\, \mathring{a} : \mathring{X}\ a_0\ a_1\,. \\
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\end{array}
$$

What do we do with $\llbracket \llbracket X \rrbracket^1 \rrbracket^2_{\bar{\rho}}$?

The problem of iterated π

$$\begin{array}{lcl} \mathcal{T} & := & \Pi\, X : \star\, .\, \Pi\, a : X\, .\, \lceil\!\lceil X \rceil\!\rceil^1\ a \\ [\![\mathcal{T}]\!]^2 & = & \lambda\, f_0\ f_1 : \mathcal{T}\, . \\ & & \quad \Pi\, X_0\ X_1 : \star\, .\, \Pi\, \mathring{X} : X_0 \to X_1 \to \star\, . \\ & & \quad \Pi\, a_0 : X_0\, .\, \Pi\, a_1 : X_1\, .\, \Pi\, \mathring{a} : \mathring{X}\ a_0\ a_1\, . \\ & & \quad [\![\lceil\!\lceil X \rceil\!\rceil^1]\!]^2_{\rho}\ a_0\ a_1\ \mathring{a}\ (f_0\ X_0\ a_0)\ (f_1\ X_1\ a_1) \end{array}$$

What do we do with $[\![\lceil\!\lceil X \rceil\!\rceil^1]\!]^2_{\rho}$?

- $[\![X]\!]^2_{\rho} = \mathring{X}$
- Bernardy and Moulin propose permuting $[\![\lceil\!\lceil X \rceil\!\rceil]\!]_{\rho} = \lceil\!\lceil [\![X]\!]_{\rho} \rceil\!\rceil$
  With a technical swapping operation that leads to the hypercubes
- But they did not consider mixed-arity internalized parametricity!
- $[\![\lceil\!\lceil X \rceil\!\rceil^1]\!]^2_{\rho} = \lceil\!\lceil [\![X]\!]^2_{\rho} \rceil\!\rceil^1 = \lceil\!\lceil \mathring{X} \rceil\!\rceil^1$ not arity-correct
- Cannot permute interpretations at different arities

Proposed Solution: *Lifting Universals*

$$\Pi\, x\langle \bar{x} \rangle : A \,.\, B$$

Reflect renamings $[x \mapsto x_0], \cdots, [x \mapsto x_{n-1}]$ into quantifier

$[x \mapsto \mathring{x}]$ kept implicit by choosing $\mathring{x} \equiv x$

Typing for lifting universals

$$\frac{\Gamma \vdash A : \star \quad \Gamma, x\langle\bar{x}\rangle : A \vdash B : \star}{\Gamma \vdash \Pi\, x\langle\bar{x}\rangle : A \,.\, B : \star}$$

Contexts contain $x\langle\bar{x}\rangle$

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$$\frac{\Gamma \vdash \Pi\, x\langle\bar{x}\rangle : A \,.\, B : \star \quad \Gamma, x\langle\bar{x}\rangle : A \vdash t : B}{\Gamma \vdash \lambda\, x\langle\bar{x}\rangle : A \,.\, t : \Pi\, x\langle\bar{x}\rangle : A \,.\, B}$$

Expected generalization

Typing for lifting universals

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Expected generalization

$$\frac{\Gamma \vdash t' : \Pi\, x\langle\bar{x}^k\rangle : A\,.\,C \quad \Gamma \vdash t\langle\bar{t}\rangle : A}{\Gamma \vdash t'\ t\langle\bar{t}^k\rangle : [t\langle\bar{t}\rangle/x]C}$$

Substitution $[t\langle\bar{t}\rangle/x]$

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Contexts contain $x\langle\bar{x}\rangle$

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Substitution $[t\langle\bar{t}\rangle/x]$

$$\frac{(\forall\, i < k.\, \Gamma \vdash t_i : [\![A]\!]_i^k) \quad \Gamma \vdash t : [\![A]\!]_k^k\ \bar{t}}{\Gamma \vdash t\langle\bar{t}^k\rangle : A}$$

Liftings

Liftings and substitution

- Instead of just $[\![A]\!]$, have:
  - $[\![A]\!]^n_n$ for arity-n relation (last renaming $[x \mapsto \mathring{x}]$)
  - $[\![A]\!]^n_i$, with $i < n$, positional meaning (renamings $[x \mapsto x_i]$)

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$$
\begin{array}{lcll}
[t\langle \bar{t} \rangle / x][\![x]\!]^k_i & = & t_i & i < k \\
[t\langle \bar{t} \rangle / x][\![x]\!]^k_k & = & t & \\
 & \cdots & &
\end{array}
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Definitional equality $\Gamma \vdash t \simeq t'$, including following:

- Write $[\![x]\!]^{\bar{k}}_{\bar{i}}$ for $[\![\cdots[\![x]\!]^{k_l}_{i_l}\cdots]\!]^{k_0}_{i_0}$
- Apply renaming $x \mapsto x_{i_j}$ when $i_j < k_j$

$$\frac{x\langle\bar{x}^{k_j}\rangle : A \in \Gamma \quad i_j < k_j}{\Gamma \vdash [\![x]\!]^{\bar{k}}_{\bar{i}} \simeq [\![x_{i_j}]\!]^{\bar{k}\setminus k_j}_{\bar{i}\setminus i_j}}\ \mathbf{L}$$

Back to iterated π

$$
\begin{array}{lcl}
\mathcal{T} & := & \Pi\, X : \star\, .\, \Pi\, a : X\, .\, [\![X]\!]^1_1\ a \\
[\![\mathcal{T}]\!]^2 & = & \lambda\, f_0\ f_1 : \mathcal{T}\, . \\
 & & \quad \Pi\, X\langle X_0\ X_1\rangle : \star\, . \\
 & & \quad \Pi\, a\langle a_0 a_1\rangle : X\, . \\
 & & \quad [\![[\![X]\!]^1_1]\!]^2_2\ a_0\ a_1\ \mathring{a}\ (f_0\ X_0\ a_0)\ (f_1\ X_1\ a_1)
\end{array}
$$

- $[\![[\![X]\!]^1_1]\!]^2_2$ is a normal form
- Type of $(f_0\ X_0\ a_0)$ is $[\![X_0]\!]^1_1\ a_0$
- That is def. eq. to $[\![[\![X]\!]^1_1]\!]^2_0\ a_0$, because context holds $X\langle X_0\ X_1\rangle$
- Rule L says we can apply renaming $X \mapsto X_0$ under $[\![\ ]\!]^2_0$

$$
[\![[\![X]\!]^1_1]\!]^2_0 \ \simeq\ [\![X_0]\!]^1_1
$$

Conclusion

- Towards mixed arity, iterated internal parametricity
- Metatheory idea:

  Girard projection, identity for Curry-style theory [Giannini et al. 1993]
- Implementation idea:

  with implicit products, Reynolds embedding can be identity, too

Thank you.

I am recruiting a postdoc at Boston College.