NbE for LNL via Adjoint Meta-modalities

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Motivation

- Is NbE for LNL an interesting result? Probably not.
- Rather, I use NbE as a test case for (mechanised) metatheory.

Normalisation by Evaluation ($\beta\eta^-$) — Model

Take a presentation of STλC using De Bruijn indices.

$$\Gamma \vDash X \times Y := \Gamma \vDash X \;\times\; \Gamma \vDash Y$$
$$\Gamma \vDash X \otimes Y := (\Gamma \vDash X \;\times\; \Gamma \vDash Y) \;\cup\; \Gamma \vdash_{\mathrm{ne}} X \otimes Y$$
$$\Gamma \vDash X \to Y := \Pi\Gamma^+.\ \Gamma^+ \overset{\ni}{\Longrightarrow} \Gamma \;\to\; \Gamma^+ \vDash X \;\to\; \Gamma^+ \vDash Y$$

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\end{aligned}$$

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\Gamma \vDash X \to Y &\coloneqq \Box \big(\vDash X \;\; \dot{\to} \;\; \vDash Y \big) \; \Gamma
\end{aligned}$$

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\vDash X \to Y &:= \Box(\vDash X \;\dot{\to}\; \vDash Y)
\end{aligned}$$

Talking about $\mathrm{Ctx} \to \mathrm{Set}$ functions (nearly presheaves).

Normalisation by Evaluation ($\beta\eta^-$) — Re-things

Cut for time:

- Reflect
- Reify
- Renaming for normal/neutral forms
- Renaming for semantic values

Normalisation by Evaluation ($\beta\eta^{-}$) — Eval

By environment-managing recursion on the term:

$$\mathrm{eval} : \Pi\{\Gamma\Delta X\}.\ \Gamma \overset{\vDash}{\Longrightarrow} \Delta \ \rightarrow\ \Delta \vdash X \ \rightarrow\ \Gamma \vDash X$$

Normalisation:

$$\mathrm{norm} : \Pi\{\Gamma X\}.\ \Gamma \vdash X \ \rightarrow\ \Gamma \vdash_{\mathrm{nf}} X$$
$$\mathrm{norm}\, M \coloneqq \mathrm{reify}\, X\ (\mathrm{eval}\, \mathrm{id}\, M)$$

where $\mathrm{id} : \Gamma \overset{\vDash}{\Longrightarrow} \Gamma$ uses $\mathrm{reflect}$

Linear/non-Linear Logic

	Intuitionistic	Linear
Ty	X, Y, $\times$, $\to$ as before, plus G.	A, B, $\otimes$, $\multimap$, F.
Ctx	Θ, Λ as before (intuitionistic only).	Γ, Δ contain both linear and intuitionistic variables.
Env	$\overset{\triangleright}{\Longrightarrow}_{\mathcal{I}}$ as before.	$\overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}}$ given inductively on the right-hand context (see later slide).

Structural Manipulations Become Meta-connectives

- Traditionally: $\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes\text{-I}$
  - Needs exchange separately.
- More amenable to abstraction:

$$\frac{\Gamma \sim \Gamma_L + \Gamma_R \qquad \Gamma_L \vdash A \qquad \Gamma_R \vdash B}{\Gamma \vdash A \otimes B}\ \otimes\text{-I}$$

where (proof-relevantly):

$$\begin{array}{rcll}
[] \sim [] & + [] & & \\
\Gamma, \mathbf{lin}\, A \sim \Gamma_L, \mathbf{lin}\, A & + \Gamma_R & \leftarrow & \Gamma \sim \Gamma_L + \Gamma_R \\
\Gamma, \mathbf{lin}\, A \sim \Gamma_L & + \Gamma_R, \mathbf{lin}\, A & \leftarrow & \Gamma \sim \Gamma_L + \Gamma_R \\
\Gamma, \mathbf{int}\, X \sim \Gamma_L, \mathbf{int}\, X & + \Gamma_R, \mathbf{int}\, X & \leftarrow & \Gamma \sim \Gamma_L + \Gamma_R
\end{array}$$

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- Said abstraction:

$$\dfrac{(-) \vdash A \quad * \quad (-) \vdash B}{(-) \vdash A \otimes B}\ \otimes\text{-I}$$

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- Said abstraction:

$$\frac{\vdash A \quad * \quad \vdash B}{\vdash A \otimes B} \otimes\text{-I}$$

LNL's Characteristic Structural Manipulations

- Traditionally:

$$\frac{\Theta \vdash_{\mathcal{I}} X}{\Theta; [] \vdash_{\mathcal{L}} \mathrm{F}X}\ \mathrm{FI} \qquad \frac{\Theta; [] \vdash_{\mathcal{L}} A}{\Theta \vdash_{\mathcal{I}} \mathrm{G}A}\ \mathrm{GI} \qquad \frac{\Theta \vdash_{\mathcal{I}} \mathrm{G}A}{\Theta; [] \vdash_{\mathcal{L}} A}\ \mathrm{GE}$$

- This time:

$$\frac{\mathrm{F}\,(\ \vdash_{\mathcal{I}} X)}{\vdash_{\mathcal{L}} \mathrm{F}X}\ \mathrm{FI} \qquad \frac{\mathrm{G}\,(\ \vdash_{\mathcal{L}} A)}{\vdash_{\mathcal{I}} \mathrm{G}A}\ \mathrm{GI} \qquad \frac{\mathrm{F}\,(\ \vdash_{\mathcal{I}} \mathrm{G}A)}{\vdash_{\mathcal{L}} A}\ \mathrm{GE}$$

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$$\frac{\vdash_{\mathcal{L}} \mathrm{F}X \qquad * \qquad \mathbf{int}\, X \vdash_{\mathcal{L}} A}{\vdash_{\mathcal{L}} A}\ \mathrm{FE}$$

LNL NbE Model

$$
\begin{aligned}
\vDash A \otimes B &\coloneqq (\vDash A \ \ast\ \vDash B) \ \dot{\cup}\ \vdash_{\mathrm{ne}} A \otimes B \\
\vDash A \multimap B &\coloneqq \square(\vDash A \ \mathrel{-\!\!\ast}\ \vDash B) \\
\vDash \mathrm{F}X &\coloneqq \mathrm{F}(\vDash X) \ \dot{\cup}\ \vdash_{\mathrm{ne}} \mathrm{F}X
\end{aligned}
$$

$$
\begin{aligned}
\vDash X \times Y &\coloneqq \vDash X \ \dot{\times}\ \vDash Y \\
\vDash X \to Y &\coloneqq \square(\vDash X \ \dot{\to}\ \vDash Y) \\
\vDash \mathrm{G}A &\coloneqq \mathrm{G}(\vDash A)
\end{aligned}
$$

Linear Environments

$$
\begin{aligned}
\overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} [] &\ \dot{\leftarrow}\ \mathrm{I} \\
\overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta, \mathbf{lin}\, A &\ \dot{\leftarrow}\ \overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta \ * \ \blacktriangleright A \\
\overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta, \mathbf{int}\, X &\ \dot{\leftarrow}\ \overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta \ * \ \mathrm{F}(\triangleright X)
\end{aligned}
$$

A property for each meta-connective:

$$
\begin{aligned}
\Gamma \overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta \ \wedge\ \Delta \sim 0 &\ \rightarrow\ \mathrm{I}\,\Gamma \\
\Gamma \overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta \ \wedge\ \Delta \sim \Delta_L + \Delta_R &\ \rightarrow\ \left(\overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta_L \ * \ \overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta_R \right) \Gamma \\
\Gamma \overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta \ \wedge\ \Delta \sim \Lambda &\ \rightarrow\ \mathrm{F} \left(\overset{\triangleright}{\Longrightarrow}_{\mathcal{I}} \Lambda \right) \Gamma \\
\Theta \overset{\triangleright}{\Longrightarrow}_{\mathcal{I}} \Lambda \ \wedge\ \Delta \sim \Lambda &\ \rightarrow\ \mathrm{G} \left(\overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta \right) \Theta
\end{aligned}
$$

LNL Eval

Recall:

$$\mathrm{GE} : \mathrm{F}\,(\vdash_{\mathcal{I}} \mathrm{G}A) \;\dot{\to}\; \vdash_{\mathcal{L}} A$$

$$\text{env-}\mathrm{F} : \Gamma \overset{\blacktriangleright,\triangleright}{\Longrightarrow}_{\mathcal{L}} \Delta \;\wedge\; \Delta \sim \Lambda \;\to\; \mathrm{F}\left(\overset{\triangleright}{\Longrightarrow}_{\mathcal{I}} \Lambda\right)\,\Gamma$$

$$\mathrm{eval} : \Gamma \overset{\vDash}{\Longrightarrow} \Delta \;\to\; \Delta \vdash A \;\to\; \Gamma \vDash A$$

Then we get:

$$\begin{aligned}&\mathrm{eval}\,\rho\;(\mathrm{GE}\;(\mathit{rel}\;\mathrm{F}\langle M\rangle)) :=\\ &\qquad \varepsilon \;\circ\; \text{map-}\mathrm{F}\left(\lambda\rho'.\;\mathrm{eval}\,\rho'\,M\right)\;\$\;\text{env-}\mathrm{F}\,(\rho, \mathit{rel})\end{aligned}$$

where $\varepsilon : \mathrm{F}(\mathrm{G}\,T) \;\dot{\to}\; T$.

Conclusion

- Context-implicit working even for quite complicated judgemental structure
- Intuitionistic fragment is easy.
- An implicit separating/non-separating logic
- Is something like this publishable? Let me know.
- `https://github.com/laMudri/lin-env/blob/main/src/Modal/LnL.agda` (∼400 SLoC)