A Type Theory for Comprehension Categories with Applications to Subtyping

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Motivation

$$\mathrm{MLTT} \xrightarrow{\text{semantics}} \begin{tikzcd} \mathcal{T} \arrow[rr, "\chi"] \arrow[dr] & & \mathcal{C}^{\to} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd} \xrightarrow{\text{language}} \mathrm{CCTT}$$

The “semantics” arrow does not use all the features of
$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\chi"] \arrow[dr] & & \mathcal{C}^{\to} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

Two options:

1. Restrict the comprehension categories: usually done
2. Make the type theory more expressive: CCTT

Preprint is available on arXiv: https://arxiv.org/abs/2503.10868

Overview

1. Design rules of a type theory which reflect the structure of comprehension categories
2. Prove soundness by giving an interpretation of the type theory in any comprehension category
3. Extend Coraglia and Emmenegger's work [CE24] by giving rules that capture coercive subtyping
4. Develop rules for Π-, Σ- and Id-types and give soundness results wrt each suitable semantic structure
5. Extend the rules with subtyping for type formers
6. Define suitable semantic structure for subtyping for each type former and show soundness wrt to these

Outline

1. Review: Comprehension Categories
2. Our Work: Core Syntax CCTT
3. CCTT Captures Subtyping
4. Extending CCTT with Subtyping for Type Formers

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Comprehension Categories

Comprehension Category [Jac93, Definition 4.1]

A *comprehension category* consists of a category $\mathcal{C}$, a (cloven) fibration $p : \mathcal{T} \to \mathcal{C}$, and a functor $\chi : \mathcal{T} \to \mathcal{C}^{\to}$ preserving cartesian arrows, such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

A comprehension category is *full* if χ is full and faithful.
A comprehension category is *split* if p is a split fibration.

Full split comprehension categories are models for MLTT.

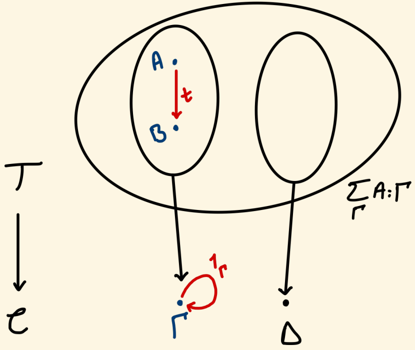
Comprehension Categories

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

1. $\mathcal{C}$: category of contexts and context morphisms
2. Fibre $\mathcal{T}_\Gamma$: category of types in context Γ
3. Substitution is captured by the reindexing functors
4. Extended context $\Gamma.A$ is given by $\mathrm{dom} \circ \chi : A \mapsto \Gamma.A$
5. $\Gamma \vdash t : A$ is interpreted as sections of $\chi(A) : \Gamma.A \to \Gamma$ in $\mathcal{C}$

Vertical Morphisms

What about morphisms in a fibre $\mathcal{T}_\Gamma$?



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CCTT: Goal

Goal: Design rules that reflect all structure of (not-necessarily full) comprehension categories.

CCTT: Judgements

1. Γ ctx
2. $\Gamma \vdash s : \Delta$
3. $\Gamma \vdash s \equiv s' : \Delta$
4. $\Gamma \vdash A$ type
5. $\Gamma | A \vdash t : B$
6. $\Gamma | A \vdash t \equiv t' : B$

CCTT: Judgements

1. Γ ctx
2. $\Gamma \vdash s : \Delta$
3. $\Gamma \vdash s \equiv s' : \Delta$
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5. $\Gamma | A \vdash t : B$
6. $\Gamma | A \vdash t \equiv t' : B$

$\left.\begin{array}{l} \text{5.} \\ \text{6.} \end{array}\right\}$ $\Gamma \vdash t : A$ & $\Gamma \vdash t \equiv t' : A$ in MLTT

CCTT: Judgements

1. Γ ctx
2. $\Gamma \vdash s : \Delta$
3. $\Gamma \vdash s \equiv s' : \Delta$
4. $\Gamma \vdash A$ type
5. $\Gamma | A \vdash t : B$
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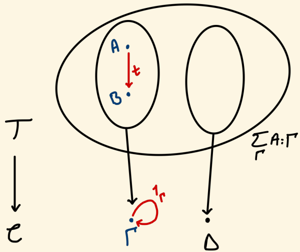

Judgement 5: a morphism $[\![t]\!] : [\![A]\!] \to [\![B]\!]$ in the fibre $\mathcal{T}_{[\![\Gamma]\!]}$.

CCTT: Structural Rules

See the paper for the structural rules.

In the next section, we discuss some rules through the lens of subtyping.

Theorem

Every comprehension category models the rules of CCTT.

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We Put Our Subtyping Glasses on

Subtyping in CCTT

Coraglia and Emmenegger [CE24] observe that the vertical morphisms can be thought of as witnesses for coercive subtyping.

$$\Gamma | A \vdash t : B \qquad \rightsquigarrow \qquad \Gamma \vdash A \leq_t B$$

Proposition (Subsumption)

From the rules of CCTT, we can derive the following rule.

$$\frac{\Gamma \vdash A, B \text{ type} \quad \Gamma \vdash A \leq_t B \quad \Gamma \vdash a : A}{\Gamma \vdash \Gamma.t \circ a : B}$$

$$\begin{tikzcd} \Gamma.A \arrow[rr, "\Gamma.t"] & & \Gamma.B \\ & \Gamma \arrow[ul, "a"] \arrow[ur, "b"'] & \end{tikzcd}$$

Γ.t is like a coercion function for $A \leq_t B$.

Subtyping: Weakening and Substitution

We have the following rule in CCTT, which corresponds to substitution for subtyping.

$$\frac{\Delta \vdash A, B \text{ type} \quad \Delta \vdash A \leq_t B \quad \Gamma \vdash s : \Delta}{\Gamma \vdash A[s] \leq_{t[s]} B[s]}$$

Proposition (Weakening for Subtyping)

From the rules of CCTT, we can derive the following rule.

$$\frac{\Gamma \vdash A, A', B \text{ type} \quad \Gamma \vdash A \leq_t A'}{\Gamma.B \vdash A[\pi_B] \leq_{t[\pi_B]} A'[\pi_{B'}]}$$

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Subtyping for Type formers

1. Extend CCTT with a type former (e.g. Σ-types) and show soundness: naturally, no rules involving judgements of the form $\Gamma \vdash A \leq_t B$ get added.

2. Extend CCTT with subtyping for the type former and show soundness: we see how through an example!

Example: Σ-types

Extend CCTT with Σ-types, e.g.:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma.A \vdash B \text{ type}}{\Gamma \vdash \Sigma_A B \text{ type}} \text{ sigma-form}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma.A \vdash B \text{ type}}{\Gamma.A.B \vdash \mathsf{pair}_{\Sigma_A B} : \Gamma.\Sigma_A B} \text{ sigma-intro}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma.A \vdash B \text{ type}}{\Gamma.\Sigma_A B \vdash \mathsf{proj}_{\Sigma_A B} : \Gamma.A.B} \text{ sigma-elim}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma.A \vdash B \text{ type}}{\begin{array}{c}\Gamma.A.B \vdash \mathsf{proj}_{\Sigma_A B} \circ \mathsf{pair}_{\Sigma_A B} \equiv 1_{\Gamma.A.B} : \Gamma.A.B \\ \Gamma.\Sigma_A B \vdash \mathsf{pair}_{\Sigma_A B} \circ \mathsf{proj}_{\Sigma_A B} \equiv 1_{\Gamma.\Sigma_A B} : \Gamma.\Sigma_A B\end{array}} \text{ sigma-beta-eta}$$

$$\frac{\Delta \vdash A \text{ type} \qquad \Delta.A \vdash B \text{ type} \qquad \Gamma \vdash s : \Delta}{\Gamma \mid \Sigma_{A[s]} B[s.A] \;\widetilde{\vdash}\; i_{\Sigma_A B, s} : (\Sigma_A B)[s]} \text{ subst-sigma}$$

Example: Subtyping for Σ-types

1. We want to have the following rule:

$$\frac{\Gamma \vdash A, A' \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma.A' \vdash B' \text{ type} \\ \Gamma \vdash A \leq_f A' \quad \Gamma.A \vdash B \leq_g B'[\Gamma.f]}{\Gamma \vdash \Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'}$$

Σ acts covariantly on both arguments.

Example: Subtyping for Σ-types

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$$\frac{\Gamma \vdash A, A' \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma.A' \vdash B' \text{ type} \qquad \Gamma \vdash A \leq_f A' \quad \Gamma.A \vdash B \leq_g B'[\Gamma.f]}{\Gamma \vdash \Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'}$$

   Σ acts covariantly on both arguments.

2. The coercion function for $\Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'$ should act as follows:

$$\Gamma.\Sigma_A B \xrightarrow{\mathsf{proj}_{\Sigma_A B}} \Gamma.A.B \xrightarrow{\chi_0 g} \Gamma.A.B'[\chi_0 f] \xrightarrow{\chi_0 f . B'} \Gamma.A'.B' \xrightarrow{\mathsf{pair}_{\Sigma_{A'} B'}} \Gamma.\Sigma_{A'} B'$$

Example: Subtyping for Σ-types

1. We want to have the following rule:

$$\frac{\Gamma \vdash A, A' \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma.A' \vdash B' \text{ type} \qquad \Gamma \vdash A \leq_f A' \quad \Gamma.A \vdash B \leq_g B'[\Gamma.f]}{\Gamma \vdash \Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'}$$

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3. Rules for functoriality for $\Sigma(-,-)$

Comprehension Categories with Subtyping for Σ-types

Definition

A comprehension category $(\mathcal{C}, \mathcal{T}, p, \chi)$ has subtyping for Σ-types if it has dependent sums and is equipped with a function giving for each $f : A \to A'$ in $\mathcal{T}_\Gamma$ and $g : B \to B'[\chi_0 f]$ in $\mathcal{T}_{\Gamma.A}$, a morphism

$$\Sigma_f g : \Sigma_A B \to \Sigma_{A'} B'$$

in $\mathcal{T}_\Gamma$ such that:

1. $\chi_0(\Sigma_f g)$ is the following composite

$$\Gamma.\Sigma_A B \xrightarrow{\mathsf{proj}_{\Sigma_A B}} \Gamma.A.B \xrightarrow{\chi_0 g} \Gamma.A'.B'[\chi_0 f] \xrightarrow{\chi_0 f.B'} \Gamma.A'.B' \xrightarrow{\mathsf{pair}_{\Sigma_{A'} B'}} \Gamma.\Sigma_{A'} B'$$

2. $\Sigma_{(-)}(-)$ preserves identities and composition

Theorem

Any comprehension category with subtyping for Σ-types models CCTT extended with subtyping for Σ-types.

Related Work

- Zeilberger and Melliès [MZ15] give a fibrational view of subsumptive subtyping
- Coraglia and Emmenegger [CE24] study type morphisms as witnesses for coercive subtyping
- Laurent, Lennon-Bertrand and Maillard [LLM24] show equivalence between their type theories with coercive and subsumptive subtyping
- Last talk today: 'AdapTT: A Type Theory with Functorial Types' by Adjedj, Benjamin, Lennon-Bertrand and Maillard

Summary

1. We presented CCTT
2. CCTT captures coercive subtyping
3. We extended CCTT with Π (resp. Σ, Id) and subtyping for Π (resp. Σ, Id)
4. At each step we showed soundness wrt to the suitable semantic structure

Thank you for your attention!

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[MZ15] Paul-André Melliès and Noam Zeilberger. "Functors are Type Refinement Systems". In: ed. by Sriram K. Rajamani and David Walker. ACM, 2015, pp. 3–16. DOI: 10.1145/2676726.2676970. URL: https://doi.org/10.1145/2676726.2676970.

Structural Rules: $\mathcal{C}$

Structural rules regarding the category of contexts

$$\frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash 1_\Gamma : \Gamma} \ \text{ctx-mor-id} \qquad \frac{\Gamma, \Delta, \Theta \ \mathsf{ctx} \qquad \Gamma \vdash s : \Delta \qquad \Delta \vdash s' : \Theta}{\Gamma \vdash s' \circ s : \Theta} \ \text{ctx-mor-comp}$$

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \qquad \Gamma \vdash s : \Delta}{\begin{array}{c}\Gamma \vdash s \circ 1_\Gamma \equiv s : \Delta \\ \Gamma \vdash 1_\Delta \circ s \equiv s : \Delta\end{array}} \ \text{ctx-id-unit}$$

$$\frac{\Gamma, \Delta, \Theta, \Phi \ \mathsf{ctx} \qquad \Gamma \vdash s : \Delta \qquad \Delta \vdash s' : \Theta \qquad \Theta \vdash s'' : \Phi}{\Gamma \vdash s'' \circ (s' \circ s) \equiv (s'' \circ s') \circ s : \Phi} \ \text{ctx-comp-assoc}$$

Structural Rules: $\mathcal{T}$

Structural rules regarding the category of types

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma \mid A \vdash 1_A : A} \text{ ty-mor-id}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B, C \text{ type} \quad \Gamma \mid A \vdash t : B \quad \Gamma \mid B \vdash t' : C}{\Gamma \mid A \vdash t' \circ t : C} \text{ ty-mor-comp}$$

$$\frac{\Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\begin{array}{c}\Gamma \mid A \vdash t \circ 1_A \equiv t : B \\ \Gamma \mid A \vdash 1_B \circ t \equiv t : B\end{array}} \text{ ty-id-unit}$$

$$\frac{\Gamma \mid A \vdash t : B \quad \Gamma \mid B \vdash t' : C \quad \Gamma \mid C \vdash t'' : D}{\Gamma \mid A \vdash t'' \circ (t' \circ t) \equiv (t'' \circ t') \circ t : D} \text{ ty-comp-assoc}$$

Structural Rules: Context Extension

Structural rules regarding context extension

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma.A \text{ ctx}} \text{ ext-ty}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\Gamma.A \vdash \Gamma.t : \Gamma.B} \text{ ext-tm}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma.A \vdash \Gamma.1_A \equiv 1_{\Gamma.A} : \Gamma.A} \text{ ext-id}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B, C \text{ type} \quad \Gamma \mid A \vdash t : B \quad \Gamma \mid B \vdash t' : C}{\Gamma.A \vdash \Gamma.(t' \circ t) \equiv \Gamma.t' \circ \Gamma.t : \Gamma.B} \text{ ext-comp}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma.A \vdash \pi_A : \Gamma} \text{ ext-proj}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t : B}{\Gamma.A \vdash \pi_B \circ \Gamma.t \equiv \pi_A : \Gamma} \text{ ext-coh}$$

Structural Rules: Substitution (Part 1)

Structural rules regarding substitution

$$\frac{\Gamma \vdash s : \Delta \qquad \Delta \vdash A \text{ type}}{\Gamma \vdash A[s] \text{ type}}\ \text{sub-ty} \qquad \frac{\Delta \vdash A, B \text{ type} \qquad \Gamma \vdash s : \Delta \qquad \Delta \mid A \vdash t : B}{\Gamma \mid A[s] \vdash t[s] : B[s]}\ \text{sub-tm}$$

$$\frac{\Gamma, \Delta \text{ ctx} \qquad \Gamma \vdash s : \Delta \qquad \Delta \vdash A \text{ type}}{\Gamma \mid A[s] \vdash 1_A[s] \equiv 1_{A[s]} : A[s]}\ \text{sub-pres-id}$$

$$\frac{\Gamma \vdash s : \Delta \qquad \Delta \vdash A, B, C \text{ type} \qquad \Delta \mid A \vdash t : B \qquad \Delta \mid B \vdash t' : C}{\Gamma \mid A[s] \vdash (t' \circ t)[s] \equiv t'[s] \circ t[s] : C[s]}\ \text{sub-pres-comp}$$

$$\frac{\Gamma \text{ ctx} \qquad \Gamma \vdash A \text{ type}}{\Gamma \mid A[1_\Gamma] \overset{\sim}{\vdash} i^{\text{id}}_A : A}\ \text{sub-id} \qquad \frac{\Gamma \vdash u : \Delta \qquad \Delta \vdash u' : \Theta \qquad \Theta \vdash A \text{ type}}{\Gamma \mid A[u' \circ u] \overset{\sim}{\vdash} i^{\text{comp}}_{A,u',u} : A[u'][u]}\ \text{sub-comp}$$

$$\frac{\Gamma \text{ ctx} \qquad \Gamma \vdash A, B \text{ type} \qquad \Gamma \mid A \vdash t : B}{\Gamma \mid A[1_\Gamma] \vdash t[1_\Gamma] \equiv {i^{\text{id}}_B}^{-1} \circ t \circ i^{\text{id}}_A : B[1_\Gamma]}\ \text{sub-tm-id}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \qquad \Theta \vdash A, B \text{ type} \qquad \Gamma \vdash u : \Delta \qquad \Delta \vdash u' : \Theta \qquad \Theta \mid A \vdash t : B}{\Gamma \mid A[u' \circ u] \vdash t[u' \circ u] \equiv {i^{\text{comp}}_{B,u',u}}^{-1} \circ t[u'][u] \circ i^{\text{comp}}_{A,u',u} : B[u' \circ u]}\ \text{sub-tm-comp}$$

Structural Rules: Substitution (Part 2)

Structural rules regarding substitution

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash t : \Gamma.A[s] \quad \Gamma \vdash \pi_{A[s]} \circ t \equiv 1_\Gamma : \Gamma}{\Gamma \vdash (s, t) : \Delta.A} \text{ sub-ext}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta.A}{\begin{array}{c}\Gamma \vdash p_2(s) : \Gamma.A[\pi_A \circ s] \\ \Gamma \vdash \pi_{A[\pi_A \circ s]} \circ p_2(s) \equiv 1_\Gamma : \Gamma\end{array}} \text{ sub-proj}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta.A}{\Gamma \vdash (\pi_A \circ s, p_2(s)) \equiv s : \Delta.A} \text{ sub-eta}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash t : \Gamma.A[s] \quad \Gamma \vdash \pi_{A[s]} \circ t \equiv 1_\Gamma : \Gamma}{\begin{array}{c}\Gamma \vdash \pi_A \circ (s, t) \equiv s : \Delta \\ \Gamma \vdash p_2(s, t) \equiv t : \Gamma.A[s]\end{array}} \text{ sub-beta}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A, B \text{ type} \quad \Delta \mid A \vdash t : B \quad \Gamma \vdash s : \Delta}{\Gamma.A[s] \vdash \Gamma.t[s] \equiv p_2(\Delta.t \circ (s \circ \pi_{A[s]}, \Gamma.i^{\text{comp}}_{A,s,\pi_{A[s]}}[\pi_{A[s]}] \circ p_2(1_{\Gamma.A[s]}))) : \Gamma.B[s]} \text{ tm-sub-coh}$$

Structural Rules: $\equiv$ (Part 1)

Structural rules regarding $\equiv$ being a congruence relation

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma \vdash s \equiv s : \Delta} \text{ ctx-eq-refl}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s_1, s_2 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta}{\Gamma \vdash s_2 \equiv s_1 : \Delta} \text{ ctx-eq-sym}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s_1, s_2, s_3 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta \quad \Gamma \vdash s_2 \equiv s_3 : \Delta}{\Gamma \vdash s_1 \equiv s_3 : \Delta} \text{ ctx-eq-trans}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash s_1, s_2 : \Delta \quad \Delta \vdash t : \Theta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta}{\Gamma \vdash t \circ s_1 \equiv t \circ s_2 : \Theta} \text{ ctx-comp-cong-1}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash t : \Delta \quad \Delta \vdash s_1, s_2 : \Theta \quad \Delta \vdash s_1 \equiv s_2 : \Theta}{\Gamma \vdash s_1 \circ t \equiv s_2 \circ t : \Theta} \text{ ctx-comp-cong-2}$$

Structural Rules: $\equiv$ (Part 2)

Structural rules regarding $\equiv$ being a congruence relation

$$\dfrac{\Gamma\ \mathsf{ctx} \qquad \Gamma \vdash A, B\ \mathsf{type} \qquad \Gamma \mid A \vdash t : B}{\Gamma \mid A \vdash t \equiv t : B}\ \mathsf{ty\text{-}eq\text{-}refl}$$

$$\dfrac{\Gamma \vdash A, B\ \mathsf{type} \qquad \Gamma \mid A \vdash t_1, t_2 : B \qquad \Gamma \mid A \vdash t_1 \equiv t_2 : B}{\Gamma \mid A \vdash t_2 \equiv t_1 : B}\ \mathsf{ty\text{-}eq\text{-}sym}$$

$$\dfrac{\Gamma \mid A \vdash t_1, t_2, t_3 : B \qquad \Gamma \mid A \vdash t_1 \equiv t_2 : B \qquad \Gamma \mid A \vdash t_2 \equiv t_3 : B}{\Gamma \mid A \vdash t_1 \equiv t_3 : B}\ \mathsf{ty\text{-}eq\text{-}trans}$$

$$\dfrac{\Gamma \mid A \vdash u_1, u_2 : B \qquad \Gamma \mid B \vdash v : C \qquad \Gamma \mid A \vdash u_1 \equiv u_2 : B}{\Gamma \mid A \vdash v \circ u_1 \equiv v \circ u_2 : C}\ \mathsf{ty\text{-}comp\text{-}cong\text{-}1}$$

$$\dfrac{\Gamma \mid A \vdash v : B \qquad \Gamma \mid B \vdash u_1, u_2 : C \qquad \Gamma \mid B \vdash u_1 \equiv u_2 : C}{\Gamma \mid A \vdash u_1 \circ v \equiv u_2 \circ v : C}\ \mathsf{ty\text{-}comp\text{-}cong\text{-}2}$$

Structural Rules: $\equiv$ (Part 3)

Structural rules regarding $\equiv$ being a congruence relation

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma \mid A \vdash t_1, t_2 : B \quad \Gamma \mid A \vdash t_1 \equiv t_2 : B}{\Gamma.A \vdash \Gamma.t_1 \equiv \Gamma.t_2 : \Gamma.B} \text{ ext-cong}$$

$$\frac{\Gamma \vdash s : \Delta \quad \Delta \mid A \vdash t_1, t_2 : B \quad \Delta \mid A \vdash t_1 \equiv t_2 : B}{\Gamma \mid A[s] \vdash t_1[s] \equiv t_2[s] : B[s]} \text{ sub-cong-tm}$$

$$\frac{\Gamma \vdash s_1, s_2 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta \quad \Gamma \vdash t_1, t_2 : A[s] \quad \Gamma \vdash t_1 \equiv t_2 : A[s]}{\Gamma \vdash (s_1, t_1) \equiv (s_2, t_2) : \Delta.A} \text{ sub-ext-cong}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s_1, s_2 : \Delta.A \quad \Gamma \vdash s_1 \equiv s_2 : \Delta.A}{\begin{array}{c}\Gamma \vdash p_1(s_1) \equiv p_1(s_2) : \Delta \\ \Gamma \vdash p_2(s_1) \equiv p_2(s_2) : A[s]\end{array}} \text{ sub-proj-cong}$$

Structural Rules: $\equiv$ (Part 4)

Structural rules regarding there being no judgement for equality of types

$$\frac{\Delta \vdash A \text{ type} \qquad \Gamma \vdash u \equiv u' : \Delta}{\begin{array}{c}\Gamma \mid A[u] \overset{\sim}{\vdash} i^{\text{sub}}_{A,u,u'} : A[u'] \\ \Gamma \mid A[u'] \vdash i^{\text{sub}^{-1}}_{A,u,u'} \equiv i^{\text{sub}}_{A,u',u} : A[u]\end{array}} \text{ sub-cong}$$

$$\frac{\Delta \vdash A \text{ type} \qquad \Gamma \vdash u : \Delta}{\Gamma \mid A[u] \vdash i^{\text{sub}}_{A,u,u} \equiv 1_{A[u]} : A[u]} \text{ sub-cong-id}$$

$$\frac{\Theta \vdash A \text{ type} \qquad \Gamma \vdash t, t' : \Delta \qquad \Delta \vdash u : \Theta \qquad \Gamma \vdash t \equiv t' : \Delta}{\Gamma \mid A[u][t] \vdash i^{\text{comp}}_{A,u,t'} \circ i^{\text{sub}}_{A,t,t'} \equiv i^{\text{sub}}_{A,u\circ t,u\circ t'} \circ i^{\text{comp}}_{A,u,t} : A[u \circ t']} \text{ sub-cong-comp-1}$$

$$\frac{\Theta \vdash A \text{ type} \qquad \Gamma \vdash t : \Delta \qquad \Delta \vdash u, u' : \Theta \qquad \Gamma \vdash u \equiv u' : \Delta}{\Gamma \mid A[u][t] \vdash i^{\text{comp}}_{A,u',t} \circ i^{\text{sub}}_{A,u,u'}[t] \equiv i^{\text{sub}}_{A,u\circ t,u'\circ t} \circ i^{\text{comp}}_{A,u,t} : A[u' \circ t]} \text{ sub-cong-comp-2}$$