HoTTLean

Formalizing the Meta-Theory of HoTT in Lean

Joseph Hua [1] with Steve Awodey[1], Mario Carneiro[2], Sina Hazratpour[3], Wojciech Nawrocki[1], Spencer Woolfson[1], and Yiming Xu[4]

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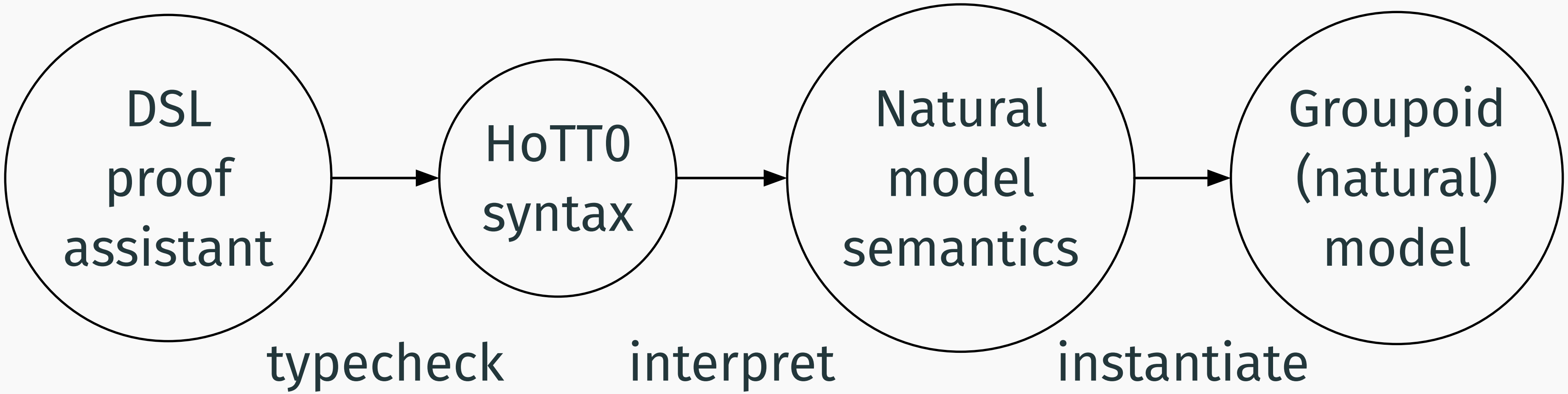
Project overview



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  - Sozeau and Tabareau 2014. Towards an internalization of the groupoid model of type theory

HoTT0 syntax

HoTT0 is a fragment of HoTT, meaning

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Noting that

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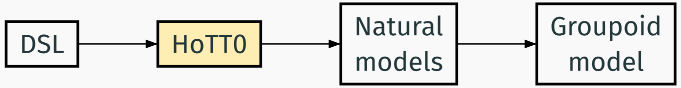

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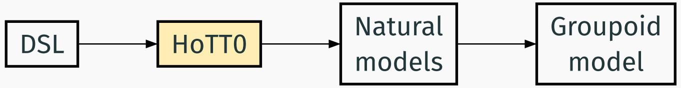

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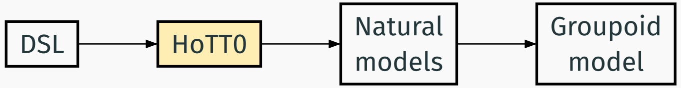

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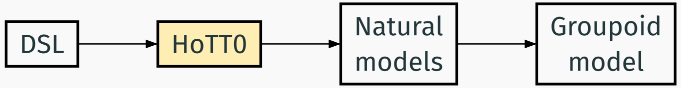

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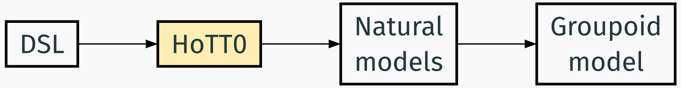

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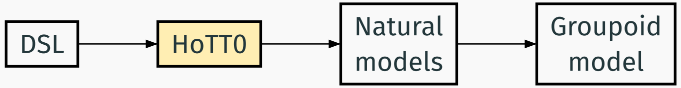
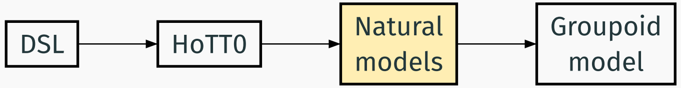
Noting that

- HoTT0 admits the **structure identity principle** for sets-level mathematics.
- Universes are not cumulative.
- Finitely many universes.



Natural model semantics - universes

In a presheaf category $\mathsf{Set}^{\mathsf{Ctx}^{\mathrm{op}}}$



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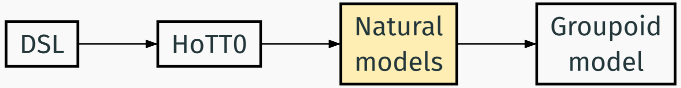

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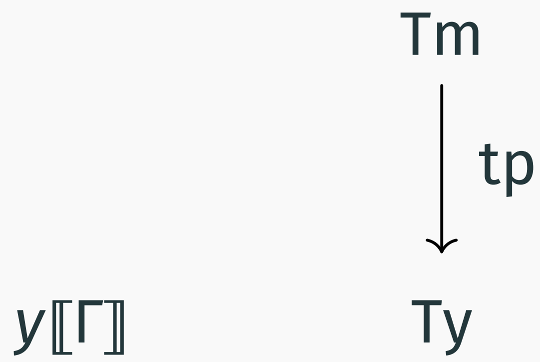

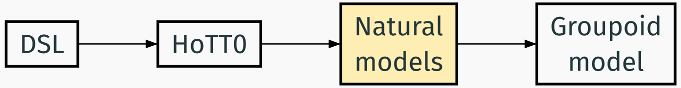

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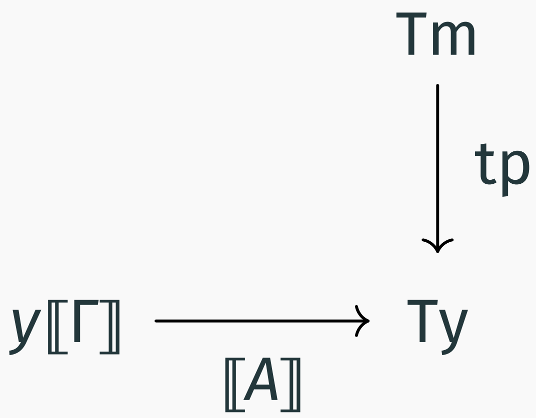

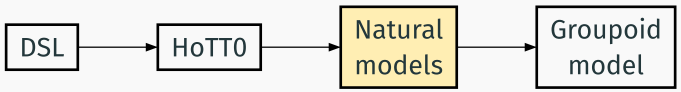

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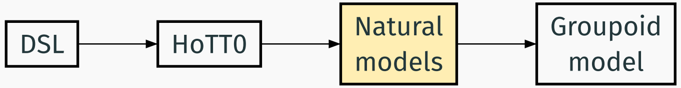
$$\begin{array}{ccc} y[\![\Gamma.A]\!] & \longrightarrow & \mathrm{Tm} \\ \downarrow & \lrcorner & \downarrow \mathrm{tp} \\ y[\![\Gamma]\!] & \xrightarrow[{[\![A]\!]}]{} & \mathrm{Ty} \end{array}$$



Natural model semantics - Π types

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\begin{array}{ccc}
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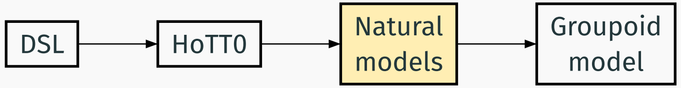

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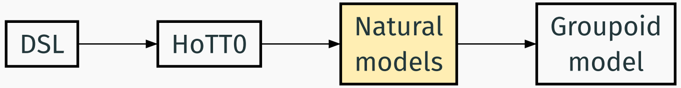

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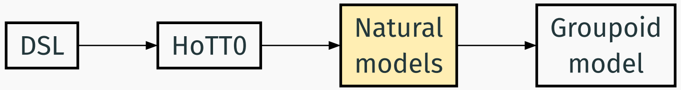

Theorems for Mathlib

- Lean formalisation of polynomial endofunctors (a.k.a containers). See github.com/sinhp/Poly project.
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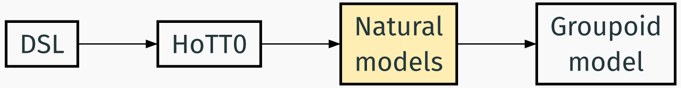

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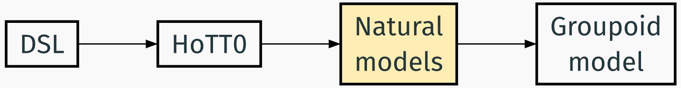

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/-- `C(Γ, PₚX) ≅ Σ(b : Γ → B), C(b*p, X)` -/
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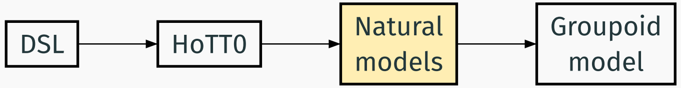

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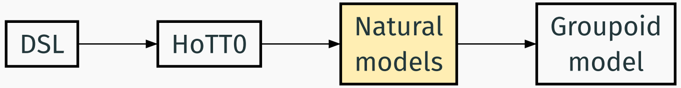
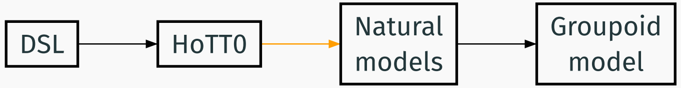
- However this currently has significant performance issues (due to heavy `rfl` proofs).



Interpretation of HoTT0 into natural model semantics

- Interpretation is a partial function on raw terms, that is defined on well-formed types and terms.



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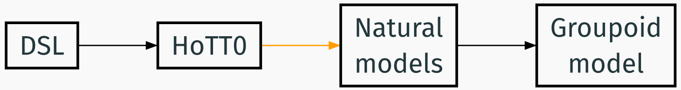

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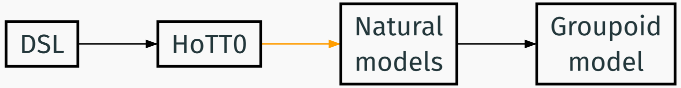
- Modular approach: we can plug in any natural model to this abstract interpretation.



Groupoid model of HoTT0

- Category of contexts Ctx = Grpd is category of "large" groupoids (with (Type 5)-sized objects and arrows).
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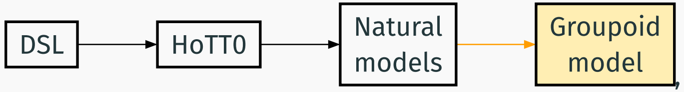

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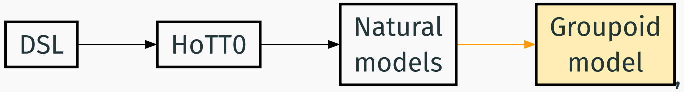

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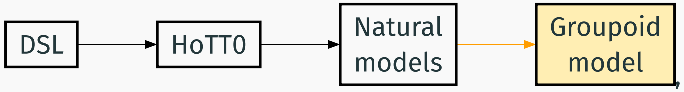
- $\mathrm{Ty}_0 = y\left(\mathrm{Grpd}_{\cong}\right)$ is (Yoneda of) the core of the category of "small" groupoids (with (Type 0)-sized objects and arrows).
- A type $A : y(\Gamma) \to \mathrm{Ty}_0$ is equivalent to a functor $\Gamma \to \mathrm{Grpd}$ from the "large" groupoid Γ into the category of "small" groupoids.



Necessary evil

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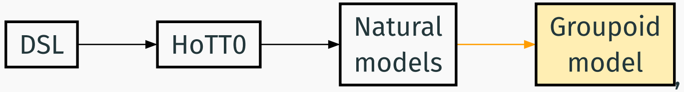

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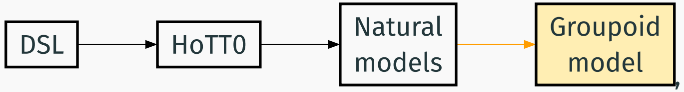

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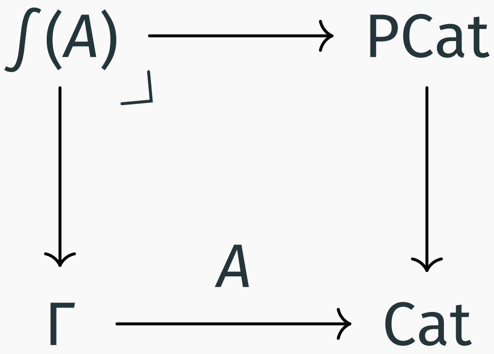

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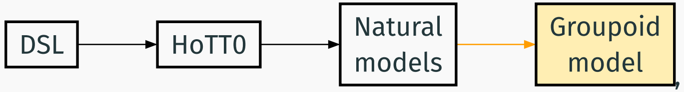

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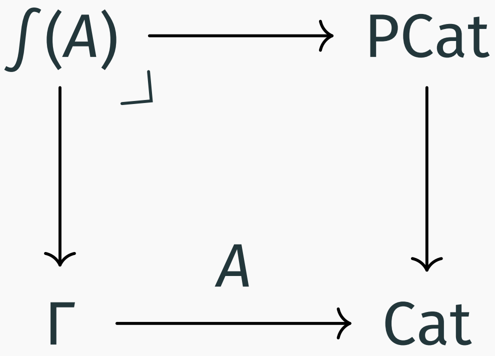
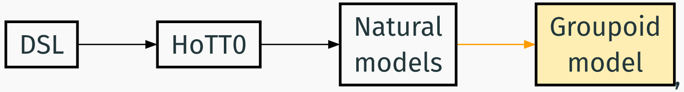


- Proven Grothendieck construction $\int(A)$ is a strict pullback of PCat. Developed API for pullbacks of categories.
- Note: Mathlib definitions are not general enough: categories in the pullback square are **not in the same category** due to universe levels.

```
Γ : Type u
Cat : Type (u+1)
```



DTT hell - rewrite along paths over paths.

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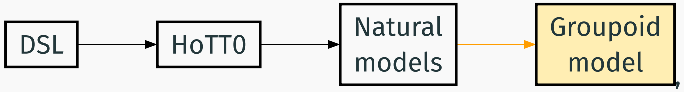

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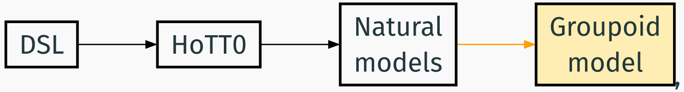

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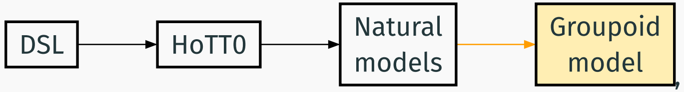

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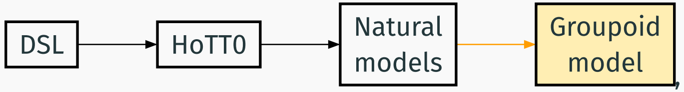
- Towards a tactic for rewriting along Lean’s heterogeneous equality `HEq`.



Domain-specific language for HoTT0

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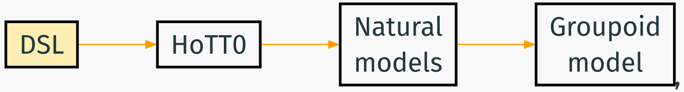

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--   tp := pi 1 0 (univ 0) (pi 0 0 (el (bvar 0)) (el (bvar 1))),
--   wf := (… : [] ⊢[l] val ≡ val : tp) }
#eval! idfun.checked -- type checker
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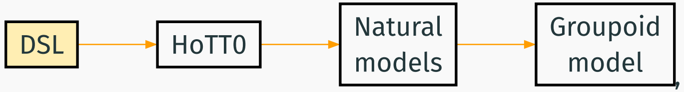

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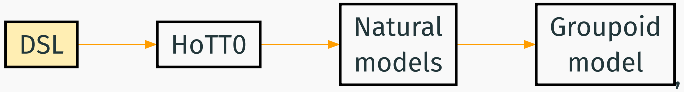
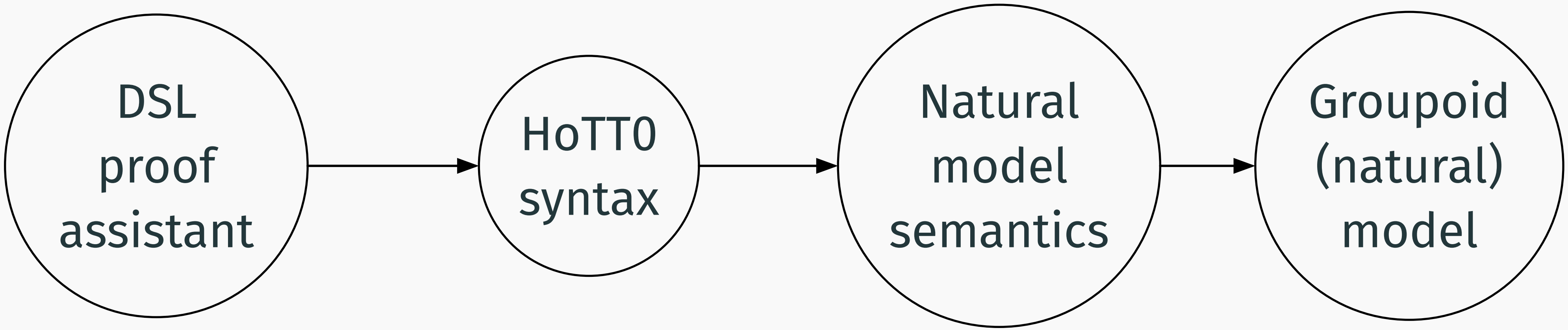
-- type interpreted in groupoid model
noncomputable def GroupoidModel.idfun.interpType :
    ⊤_ _ ⟶ GroupoidModel.Ctx.ofCategory.{1,4} Grpd.{1,1} :=
  (uHomSeqPis.interpType …
    idfun.checked.wf.wf_tp … uHomSeqPis.nilCObj …).app (.op ◁ ⊤_ _)
(𝟙 _)
```



Project progress



typechecker	$U_n, \Sigma, \Pi, \mathrm{Id}$	$U_n, \Sigma, \Pi, \mathrm{Id}$	$U_n, \Sigma, \Pi, \mathrm{Id}$

\+ Set Univalence + Function Extensionality

Next steps and ways to contribute

Repository: github.com/sinhp/groupoid_model_in_lean4

Possible ways to contribute:

Next steps and ways to contribute

Repository: github.com/sinhp/groupoid_model_in_lean4

Possible ways to contribute:

- Polynomial functor library
-
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Next steps and ways to contribute

Repository: github.com/sinhp/groupoid_model_in_lean4

Possible ways to contribute:

- Polynomial functor library
- Rewriting tactics (e.g. `HEq`)
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Next steps and ways to contribute

Repository: github.com/sinhp/groupoid_model_in_lean4

Possible ways to contribute:

- Polynomial functor library
- Rewriting tactics (e.g. `HEq`)
- Formalisation of identity types
-
-

Next steps and ways to contribute

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Possible ways to contribute:

- Polynomial functor library
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- User interface for DSLs
-

Next steps and ways to contribute

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Possible ways to contribute:

- Polynomial functor library
- Rewriting tactics (e.g. `HEq`)
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- Profunctor library performance issues

Next steps and ways to contribute

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LMU student thesis project.

Bibliography

[1] M. Sozeau and N. Tabareau, “Towards an internalization of the groupoid model of type theory,” *Types for Proofs and Programs 20th Meething (TYPES 2014), Book of Abstracts*, 2014.

[2] T. M. Community, “The Lean mathematical library,” in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, in CPP 2020. New Orleans, LA, USA: Association for Computing Machinery, 2020, pp. 367–381. doi: 10.1145/3372885.3373824.

[3] T. M. Community, “Mathlib.”

[4] T. Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: \url{https://homotopytypetheory.org/book}, 2013.

[5] M. Hofmann and T. Streicher, “The Groupoid Interpretation of Type Theory,” *Twenty-five years of constructive type theory (Venice, 1995)*, vol. 36. in Oxford Logic Guides, vol. 36. Oxford Univ. Press, New York, pp. 83–111, 1998.

[6] S. Ullrich and L. de Moura, “Beyond Notations: Hygienic Macro Expansion for Theorem Proving Languages,” in *Automated Reasoning*, N. Peltier and V. Sofronie-Stokkermans, Eds., Cham: Springer International Publishing, 2020, pp. 167–182.

[7] S. Awodey, “Natural models of homotopy type theory,” *Mathematical Structures in Computer Science*, vol. 28, no. 2, pp. 241–286, 2018.

[8] N. Veltri and N. van der Weide, “Constructing Higher Inductive Types as Groupoid Quotients,” *Logical Methods in Computer Science*, Apr. 2021, doi: 10.23638/lmcs-17(2:8)2021.

[9] T. Altenkirch, P. Capriotti, G. Dijkstra, N. Kraus, and F. Nordvall Forsberg, "Quotient Inductive-Inductive Types," in *Foundations of Software Science and Computation Structures*, Springer International Publishing, 2018, pp. 293–310. doi: 10.1007/978-3-319-89366-2_16.

[10] J. Vidmar, "Polynomial functors and W-types for groupoids," 2018.

[11] P. R. North, "Towards a directed homotopy type theory," *Electronic Notes in Theoretical Computer Science*, vol. 347, pp. 223–239, 2019.

[12] B. Ahrens, P. R. North, and N. van der Weide, "Bicategorical type theory: semantics and syntax," *Mathematical Structures in Computer Science*, vol. 33, no. 10, pp. 868–912, 2023.

[13] P. Aczel, "On Voevodsky's univalence axiom," *Mathematical Logic: Proof Theory, Constructive Mathematics, Samuel R. Buss, Ulrich Kohlenbach, and Michael Rathjen (Eds.). Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach*, p. 2967, 2011.

[14] J. M. Han and F. van Doorn, "A formal proof of the independence of the continuum hypothesis," in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2020, New Orleans, LA, USA, January 20-21, 2020*, 2020.

[15] Y. Xu and K. Maillard, "Geometric Reasoning in Lean: from Algebraic Structures to Presheaves," in *31st International Conference on Types for Proofs and Programs*, 2025. [Online]. Available: https://openreview.net/forum?id=H6crYKvzHI

[16] M. Carneiro and E. Riehl, "Formalizing colimits in Cat," *arXiv preprint arXiv:2503.20704*, 2025.