Directed equality with dinaturality

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\text{Types} &\rightsquigarrow \text{Categories} \\
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\text{Equalities } e : a = b &\rightsquigarrow \text{Morphisms } e : \hom(a, b) \\
=_A : A \times A \to \text{Type} &\rightsquigarrow \hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}
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$$

$\to$ Now types have a *polarity*, $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$, i.e., the opposite category.
$\to$ Now equalities $e : \hom(a, b)$ have *directionality.*

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- *Problem:* rule is not functorial w.r.t. variance of $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$, since $a : \mathbb{C}$ appears both contravariantly and covariantly.

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  $\to$ Use the maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ to collapse the two variances.
- Then a J-like rule is validated, but *again using groupoidal structure*.

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- Dinaturality solves the variance issue without groupoids, and tells what syntactic restriction to put on J to avoid symmetry.
- We give "logical rules" to (co)ends as the *directed quantifiers* of DTT: $\rightsquigarrow$ rules of DTT give *simple proofs* in category theory, with $\hom$ as $=$.
- We do first-order because (co)end calculus is typically first-order.

Syntax – judgements for types

- Judgement $\boxed{C \text{ type}}$ for types:

$$\frac{C \text{ type}}{C^{\mathsf{op}} \text{ type}} \qquad \frac{C \text{ type} \qquad D \text{ type}}{C \times D \text{ type}} \qquad \frac{C \text{ type} \qquad D \text{ type}}{[C, D] \text{ type}} \qquad \frac{}{\top \text{ type}}$$

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$$[\![\Gamma := [C_1, \ldots C_n]]\!] := [\![C_1]\!] \times \cdots \times [\![C_n]\!]$$

Directed type theory: judgements for terms

- A judgement $\boxed{\Gamma \vdash t : C}$ for simply-typed terms.
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$$\frac{[\Gamma, x : C]\ P(x)\ \mathsf{prop}}{[\Gamma]\ \int^{x:C} P(x)\ \mathsf{prop}} \quad \frac{[\Gamma, x : C]\ P(x)\ \mathsf{prop}}{[\Gamma]\ \int_{x:C} P(x)\ \mathsf{prop}}$$

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- **Semantics:** $\times$ is the pointwise product of dipresheaves in **Set**, $\Rightarrow$ is the pointwise $\hom$ in **Set**, (co)ends are always taken in **Set**.

Syntax – predicates (contd.)

- Directed equality predicates:

$$\frac{\Gamma^{\mathsf{op}}, \Gamma \vdash s : C^{\mathsf{op}} \qquad \Gamma^{\mathsf{op}}, \Gamma \vdash t : C}{[\Gamma] \;\; \hom_C(s,t) \ \mathsf{prop}}$$

- **Key idea:** I can use variables from Γ or from Γ^{op} in the terms s, t.

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- This is what allows us to *write* these entailments:

$$\begin{array}{rrl} [x : C] & \Phi \vdash \mathsf{refl} & : \hom(\overline{x}, x) \\ [a : C^{\mathsf{op}}, b : C, c : C] \; \hom(a,b), & \hom(\overline{b}, c), \Phi \vdash \mathsf{trans} & : \hom(a,c) \\ [a : C^{\mathsf{op}}, b : C] & \hom(a,b), \Phi \vdash \mathsf{sym} & : \hom(\overline{b}, \overline{a}) \end{array}$$

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- **Polarity of a position**: *positive* when taken from Γ, *negative* when Γ^{op}.
- **Variance of a variable**:
  natural when always taken from Γ,

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\end{array}$$

- **Polarity of a position**: *positive* when taken from Γ, *negative* when Γ^{op}.
- **Variance of a variable**:
  natural when always taken from Γ,
  dinatural (i.e., mixed-variance) when sometimes from Γ, sometimes Γ^{op}.

Syntax – entailments

- A judgement $\boxed{[\Gamma]\ \Phi \vdash \alpha : P}$ for entailments (Φ is a list of predicates).

$$[x : C, y : D, \Gamma]\ \Phi(\overline{x}, x, \overline{y}, y, \dots) \vdash \alpha : P(\overline{x}, x, \overline{y}, y, \dots)$$

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- **Semantics:** interpreted as dinatural transformations $[\![\alpha]\!] : [\![\Phi]\!] \xrightarrow{\bullet\bullet} [\![P]\!]$:

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- Dinaturals do not always compose; they do with *natural* transformations.

$$\frac{P \longrightarrow Q \xrightarrow{\bullet\bullet} R \longrightarrow T}{P \xrightarrow{\bullet\bullet} T}$$

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- Dinaturals do not always compose; they do with *natural* transformations.

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- We capture left/right cut rules with naturals, e.g.: nat on the right:

$$\frac{\begin{array}{r} P, Q \text{ do not depend on } \Gamma \\ [z : C, \Gamma]\ \Phi(\overline{z}, z) \vdash \gamma \quad\quad : P(\overline{z}, z) \\ [a : C^{\mathsf{op}}, b : C, \Gamma]\ k : P(a, b), \Phi(\overline{a}, \overline{b}) \vdash \alpha[k] : Q(a, b) \end{array}}{[z : C, \Gamma]\ \Phi(\overline{z}, z) \vdash \alpha[\gamma] : Q(\overline{z}, z)}\ \text{(cut-nat)}$$

Syntax – entailments

- A judgement $\boxed{[\Gamma]\ \Phi \vdash \alpha : P}$ for entailments (Φ is a list of predicates).

$$[x : C, y : D, \Gamma]\ \Phi(\overline{x}, x, \overline{y}, y, \dots) \vdash \alpha : P(\overline{x}, x, \overline{y}, y, \dots)$$

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Takeaway: whenever we need dinats to compose, they do because of this.

Syntax – rules for $\hom$

- Directed equality introduction:

$$\frac{}{[x : C, \Gamma]\ \Phi \vdash \mathsf{refl}_x : \hom_C(\overline{x}, x)}\ (\mathsf{refl})$$

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- **Semantics:** functoriality of $[\![\Phi]\!]$ and $[\![P]\!]$.

Directed type theory with dinaturality – examples

Example (Transitivity of directed equality)

Composition is natural in $a : C^{\mathsf{op}}, c : C$ and dinatural in $b : C$:

$$\dfrac{\dfrac{}{[z : C, c : C] \qquad g : \hom(\overline{z}, c) \vdash g : \hom(\overline{z}, c)}\ (\mathsf{var})}{[a : C^{\mathsf{op}}, b : C, c : C]\ f : \hom(a, b),\ g : \hom(\overline{b}, c) \vdash J(g) : \hom(a, c)}\ (J)$$

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We contract $f : \hom(a, b)$. Rule (J) can be applied: a, b appear only negatively in ctx (a does not) and positively in conclusion ($\overline{b}$ does not).

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Functoriality of terms P is natural in $a : C^{\mathsf{op}}, b : C$ for terms $C \vdash F : D$:

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Example (Transport)

Functoriality of predicates P is natural in $b : C$, dinatural in $a : C$:

$$\dfrac{\dfrac{}{[z : C] \ p : P(z) \vdash p : P(z)}\ (\mathsf{var})}{[a : C^{\mathsf{op}}, b : C] \ e : \hom(a, b), p : P(\overline{a}) \vdash J(p) : P(b)}\ (J)$$

Directed type theory with dinaturality – non-examples

Failure of symmetry for directed equality

The restrictions do *not* allow us to obtain directed equality is symmetric:

$$[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ e : \hom(a, b) \nvdash \mathsf{sym} : \hom(\bar{b}, \overline{a})$$

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- Semantically, the interval $I := \{0 \to 1\}$ is a counterexample to derivability of this entailment in the syntax.

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Example (Left unitality for composition)

$$\frac{}{[z : C, c : C]\ g : \hom(\overline{z}, c) \vdash \mathsf{comp}[\mathsf{refl}_z, g] = g : \hom(\overline{z}, c)}\ (J\text{-comp})$$

Example (Terms send identities to identities)

$$\frac{}{[z : C]\ \Phi \vdash \mathsf{map}[\mathsf{refl}_z] = F^*(\mathsf{refl}_z) : \hom(F(\overline{z}), F(z))}\ (J\text{-comp})$$

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To prove associativity, simply contract $f : \hom(a, b)$:

$$\frac{\dfrac{}{[z, c, d : C]\quad g : \hom(\overline{z}, c), h : \hom(\overline{c}, d) \vdash \mathsf{refl}_z \mathbin{;} (g \mathbin{;} h) = (\mathsf{refl}_z \mathbin{;} g) \mathbin{;} h : \hom(\overline{z}, d)}\ (J\text{-comp})}{[a, b, c, d : C]\ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c), h : \hom(\overline{c}, d) \vdash f \mathbin{;} (g \mathbin{;} h) = (f \mathbin{;} g) \mathbin{;} h : \hom(\overline{a}, d)}\ (J\text{-eq})$$

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- This also works for dinaturality because *transport is a natural.*

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Example (Natural transformations for terms)

Given a natural transformation α from F to G,

$$\frac{}{[x : C] \ \cdot \vdash \alpha : \hom_D(F(\overline{x}), G(x))}$$

We prove naturality of families simply by contracting $f : \hom(a, b)$:

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- We can internalize all these transformations using ends:

$$[\,] \ \cdot \vdash \alpha : \mathsf{Nat}(F, G) := \int_{x:C} \hom_D(F(\overline{x}), G(x))$$

$$[\,] \ \cdot \vdash \alpha : \mathsf{Nat}(P, Q) := \int_{x:C} P(\overline{x}) \Rightarrow Q(x)$$

Directed type theory: logical rules

- Logical rules are given as isomorphisms in "adjoint form":

$$\frac{[\Gamma]\ \Phi \vdash P \times Q}{[\Gamma]\ \Phi \vdash P, \qquad [\Gamma]\ \Phi \vdash Q}\ \text{(prod)}$$

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- Rules for (co)ends in "adjoint" form:

$$\frac{[a : C, \Gamma]\ \Phi \vdash Q(\overline{a}, a)}{[\Gamma]\ \Phi \vdash \int_{a:C} Q(\overline{a}, a)}\ \text{(end)} \qquad \frac{[\Gamma]\ \left(\int^{a:C} Q(\overline{a}, a)\right), \Phi \vdash P}{[a : C, \Gamma]\ Q(\overline{a}, a), \Phi \vdash P}\ \text{(coend)}$$

- This is the presentation $\forall/\exists$-as-adjoints, up to composition of dinaturals.

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Theorem

*There is a bijection (natural in $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to$ **Set**) between sets of dinaturals and sets of **naturals** like this:*

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- **Syntax:** all rules for hom are derivable $\iff$ (J) is an iso is derivable.

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(Co)end calculus

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- We use (co)end calculus-style reasoning, i.e., we show that two presheaves are isomorphic using Yoneda.
- Adjoint form is better suited to (co)end calculus style reasoning: term-based reasoning is hard because of dinaturality.
- Rules for (co)ends as quantifiers + directed equality:
  - (Co)Yoneda,
  - Adjointess of Kan extensions via (co)ends,
  - Presheaves are closed under exponentials,
  - Associativity of composition of profunctors,
  - Right lifts in profunctors,
  - (Co)ends preserve limits,
  - Adjointness of (co)ends in natural transformations,
  - Characterization of dinaturals as certain ends,
  - Frobenius property of (co)ends using exponentials.

(Co)end calculus with dinaturality (1)

Yoneda lemma: $(\llbracket P \rrbracket, \llbracket \Gamma \rrbracket : \llbracket C \rrbracket \to \textbf{Set})$

$$\dfrac{\dfrac{\dfrac{[a : C]\ \Gamma(a) \vdash \displaystyle\int_{x:C} \hom_C(a, \overline{x}) \Rightarrow P(x)}{[a : C, x : C]\ \Gamma(a) \vdash \hom_C(a, \overline{x}) \Rightarrow P(x)}\ \text{(end)}}{[a : C, x : C]\ \ \hom_C(\overline{a}, x) \times \Gamma(a) \vdash P(x)}\ \text{(exp)}}{[z : C]\ \Gamma(z) \vdash P(z)}\ (\hom)$$

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$$

CoYoneda lemma:

$$
\dfrac{\dfrac{[a : C]\ \int^{x:C} \hom_C(\overline{x}, a) \times P(x) \vdash \Gamma(a)}{[a : C, x : C]\ \ \hom_C(\overline{a}, x) \times P(a) \vdash \Gamma(x)}\ \text{(coend)}}{[z : C]\ P(z) \vdash \Gamma(z)}\ (\mathrm{hom})
$$

(Co)end calculus with dinaturality (2)

Presheaves are cartesian closed: ($[\![\Gamma]\!], [\![A]\!], [\![B]\!] : [\![C]\!] \to$ **Set**)

$$
\begin{aligned}
[x : C]\ \Gamma(x) &\vdash (A \Rightarrow B)(x) \\
&:= \mathsf{Nat}(\hom_C(x,-) \times A, B) \\
&\cong \int_{y:C} \hom_C(x,\overline{y}) \times A(\overline{y}) \Rightarrow B(y)
\end{aligned}
$$

$$
\frac{\dfrac{\dfrac{\dfrac{[x : C]\ \Gamma(x) \vdash \displaystyle\int_{y:C} \hom_C(x,\overline{y}) \times A(\overline{y}) \Rightarrow B(y)}{[x : C, y : C]\ \Gamma(x) \vdash \hom_C(x,\overline{y}) \times A(\overline{y}) \Rightarrow B(y)}\ \text{(end)}}{[x : C, y : C]\ A(y) \times \hom_C(\overline{x}, y) \times \Gamma(x) \vdash B(y)}\ \text{(exp)}}{[y : C]\ A(y) \times \left(\displaystyle\int^{x:C} \hom_C(\overline{x}, y) \times \Gamma(x)\right) \vdash B(y)}\ \text{(coend)}}{[y : C]\ A(y) \times \Gamma(y) \vdash B(y)}\ \text{(coYoneda)}
$$

Right Kan extensions via ends are right adjoints to precomposition with $F : C \to D$ ($P : C \to \textbf{Set}, \Gamma : D \to \textbf{Set}$):

$$
\begin{array}{cl}
[y : D]\ \Gamma(y) \vdash (\mathsf{Ran}_F P)(y) := \displaystyle\int_{x:C} \hom_D(y, F(\overline{x})) \Rightarrow P(x) & \\
\hline\hline
[x : C, y : D]\ \Gamma(y) \vdash \hom_D(y, F(\overline{x})) \Rightarrow P(x) & (\text{end}) \\
\hline\hline
[x : C, y : D]\ \hom_D(\overline{y}, F(x)) \times \Gamma(y) \vdash P(x) & (\text{exp}) \\
\hline\hline
[x : C]\ \displaystyle\int^{y:D} \hom_D(\overline{y}, F(x)) \times \Gamma(y) \vdash P(x) & (\text{coend}) \\
\hline\hline
[x : C]\ \Gamma(F(x)) \vdash P(x) & (\text{coYoneda})
\end{array}
$$

(Co)end calculus with dinaturality (5)

Fubini for ends (Γ : [] prop, $P : [C, D]$ prop)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\,]\ \Gamma \vdash \int_{x:C} \int_{y:D} P(\overline{x}, x, \overline{y}, y)}{[x : C]\ \Gamma \vdash \int_{y:D} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[x : C, y : D]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[y : D, x : C]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}\ \text{(structural property)}
$$

$$
\dfrac{\dfrac{[y : D, x : C]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y)}{[y : D]\ \Gamma \vdash \int_{x:C} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}}{[\,]\ \Gamma \vdash \int_{y:D} \int_{x:C} P(\overline{x}, x, \overline{y}, y)}\ \text{(end)}
$$

Conclusion and future work

*We have seen how dinaturality allows us to give a semantic interpretation to a first-order directed type theory in **Cat** with quantifiers, where directed equality is given by* $\hom$*-functors and quantifiers by (co)ends.*

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3. Immediate future: a working notion of *dinatural context extension*
   $\rightsquigarrow$ towards *dependent dinatural directed type theory*.

The $\int$.

Paper: *"Directed equality with dinaturality"* (`arXiv:2409.10237`)
Website: `iwilare.com` ($\leftarrow$ updated version is here!)

Thank you for the attention!