A Curry-Howard correspondence for intuitionistic inquisitive logic

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Table of Contents

1 Introduction

2 Background

3 Motivation

4 Intuitionistic inquisitive logic (InqIL)

5 A Curry-Howard correspondence for InqIL

6 Final remarks

Introduction

Outline

1 Introduction

2 Background

3 Motivation

4 Intuitionistic inquisitive logic (InqIL)

5 A Curry-Howard correspondence for InqIL

6 Final remarks

Introduction

- we introduce a typed ND for intuitionistic inquisitive logic (InqIL), including its extended variant (InqIL$^{\circ}$) with the presupposition modality $\circ$
- the term calculus we use is lambda calculus extended with a new construct select corresponding to the Split rule
- this corroborates previous observations that questions have constructive content

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- we introduce a typed ND for intuitionistic inquisitive logic (**InqIL**), including its extended variant (**InqIL$^\circ$**) with the presupposition modality $\circ$
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Outline

1 Introduction

2 Background

3 Motivation

4 Intuitionistic inquisitive logic (InqIL)

5 A Curry-Howard correspondence for InqIL

6 Final remarks

Background

- inquisitive logic is a framework for handling both statements and questions
  - incl. applications to linguistics or philosophy of language [Ciardelli, 2023, Ciardelli et al., 2013]
- it is well-explored from model-theoretic and algebraic points of view [Roelofsen, 2013, Ciardelli et al., 2019].
  - recently, there has been progress in the proof-theoretic investigation [Stafford, 2021, Müller, 2023]
- however, when it comes to a type-theoretic view, the picture of inquisitive logic becomes less clear
  - to our knowledge, this area has not yet been explored

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Outline

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2. Background
3. Motivation
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6. Final remarks

Motivation

> [inquisitive] proofs have an interesting kind of constructive content, reminiscent of the proofs-as-programs interpretation of intuitionistic logic ([Ciardelli, 2023], p. 3)

- prop-as-information types vs. prop-as-types interpretation
- resolution vs. BHK clauses

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Outline

1 Introduction

2 Background

3 Motivation

4 Intuitionistic inquisitive logic (InqIL)

5 A Curry-Howard correspondence for InqIL

6 Final remarks

Intuitionistic inquisitive logic (InqIL)

Language of InqIL

Formulas:

$$\varphi, \psi ::= p \mid \bot \mid \varphi \to \psi \mid \varphi \land \psi \mid \varphi \lor \psi$$

Defined connectives:

$\neg\varphi =_{df} \varphi \to \bot$

$\varphi \equiv \psi =_{df} (\varphi \to \psi) \land (\psi \to \varphi)$

Declarative formulas:

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Intuitionistic inquisitive logic (InqIL)

Rules of InqIL

IPC + Split ([Ciardelli et al., 2020, Punčochář, 2016]):

$$\frac{\alpha \to (\varphi \vee \psi)}{(\alpha \to \varphi) \vee (\alpha \to \psi)}\ \text{Split}$$

a generalization of Kreisel-Putnam/Harrop rule:

$$\frac{\neg\chi \to (\varphi \vee \psi)}{(\neg\chi \to \varphi) \vee (\neg\chi \to \psi)}$$

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3 Motivation

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6 Final remarks

A Curry-Howard correspondence for InqIL

There are two main tasks:

1. how to deal with the Split rule
2. how to deal with declarative formulas

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1. Split rule

$$\frac{\alpha \to (\varphi \vee \psi)}{(\alpha \to \varphi) \vee (\alpha \to \psi)}$$

Variant A:

$$\frac{f : \alpha \to (\varphi \vee \psi)}{\mathsf{split}(f) : (\alpha \to \varphi) \vee (\alpha \to \psi)}\ \text{Split}$$

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1. Split rule

$$\dfrac{\begin{matrix}[\alpha]^k \\ \vdots \\ \varphi \vee \psi\end{matrix} \qquad \begin{matrix}[\alpha \to \varphi]^i \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\alpha \to \psi]^j \\ \vdots \\ \chi\end{matrix}}{\chi}\ \mathsf{S}_{i,j,k}$$

Variant B:

$$\dfrac{\begin{matrix}[z : \alpha]^k \\ \vdots \\ t : \varphi \vee \psi\end{matrix} \qquad \begin{matrix}[x : \alpha \to \varphi]^i \\ \vdots \\ u(x) : \chi\end{matrix} \qquad \begin{matrix}[y : \alpha \to \psi]^j \\ \vdots \\ v(y) : \chi\end{matrix}}{\mathsf{select}(z.t, x.u, y.v) : \chi}\ \mathsf{S}_{i,j,k}$$

How to evaluate select?

A Curry-Howard correspondence for InqIL

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How to evaluate select?

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2. Declarative formulas

We switch from declarative formulas to Harrop formulas.

Harrop formulas:

$$\delta ::= p \mid \bot \mid \varphi \to \delta \mid \delta \wedge \delta$$

Prop. [Ferguson and Punčochář, 2025]

For every Harrop formula δ there is an equivalent $\vee$-free formula α.

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Open terms theorem ([Smith, 1993])

For any term $t(z_1, \ldots, z_n)$ of type φ with free variables $z_1, \ldots, z_n$ ranging over types $\delta_1, \ldots, \delta_n$, there is a canonical form $can(z_1, \ldots, z_n)$ such that

$$t(c(z_1, \ldots, z_n)) \Longrightarrow can(z_1, \ldots, z_n)$$

where $c(z_1, \ldots, z_n)$ can be recursively constructed out of $z : C$.

(also [Goad, 1980])

A Curry-Howard correspondence for InqIL

Variant B+ ([Pezlar, 2024]):

$$\dfrac{\begin{matrix}[z:\delta]^k \\ \vdots \\ t:\varphi\vee\psi\end{matrix} \qquad \begin{matrix}[x:\delta\to\varphi]^i \\ \vdots \\ u(x):\chi\end{matrix} \qquad \begin{matrix}[y:\delta\to\psi]^j \\ \vdots \\ v(y):\chi\end{matrix}}{\mathsf{select}(z.t, x.u, y.v):\chi}\ \mathsf{S}_{i,j,k}$$

Computation rules:

$$\mathsf{select}(x.\mathsf{inl}(t_1(x)), x.u(x), y.v(y)) \Longrightarrow u(\lambda x.t_1(x))$$
$$\mathsf{select}(x.\mathsf{inr}(t_2(x)), x.u(x), y.v(y)) \Longrightarrow v(\lambda x.t_2(x))$$

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Adding a presupposition modality

- presuppositions = informative content of questions (non-inquisitive closure)
- we capture it via a modality $\circ$ that turns (inquisitive) formulas into declarative ones ([Punčochář and Pezlar, 2024])
  - inspired by truncation from HoTT

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Formulas:

$$\varphi, \psi ::= p \mid \bot \mid \varphi \to \psi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \circ\varphi$$

Term language:

$$\begin{aligned} t, s, u ::= {} & x, y, \ldots \\ & \mid \lambda x.t \mid \mathsf{ap}(t, s) \\ & \mid \langle t, s \rangle \mid \mathsf{fst}(t) \mid \mathsf{snd}(t) \\ & \mid \mathsf{inl}(t) \mid \mathsf{inr}(t) \mid \mathsf{select}(z.c, x.d, y.e) \\ & \mid \mathsf{pre}(t) \mid \mathsf{sup}(s, x.u) \end{aligned}$$

A Curry-Howard correspondence for InqIL

Introduction and elimination rules:

$$\frac{t : \varphi}{\mathsf{pre}(t) : \circ\varphi}\ \circ\mathsf{I} \qquad \frac{s : \circ\varphi \qquad \begin{array}{c}[x : \varphi]^i \\ \vdots \\ h(x) : \delta\end{array}}{\mathsf{sup}(s, x.h) : \delta}\ \circ\mathsf{E}_i$$

Computation rule:

$$\mathsf{sup}(\mathsf{pre}(t), x.h) \Longrightarrow h(t)$$

- InqIL + ∘I/E = InqIL°
  - declarative ≠ Harrop formulas

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Outline

1 Introduction

2 Background

3 Motivation

4 Intuitionistic inquisitive logic (InqIL)

5 A Curry-Howard correspondence for InqIL

6 Final remarks

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- InqIL is an intermediate logic
  - constructivity beyond intuitionistic logic
  - normalization property, disjunction property
- future work:
  - fully schematic variant
  - unrestricted variant
  - first-order variant

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Final remarks

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Final remarks

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