Lightweight Agda Formalization of Denotational Semantics

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About the topic

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Formalization

- of (new or existing) ***mathematical*** definitions

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Formalization

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Denotational semantics

- with ***recursively-defined Scott-domains***, ***fixed points***, ***λ-notation***

Original motivation

A Denotational Semantics of Inheritance and its Correctness

(1963–2021)

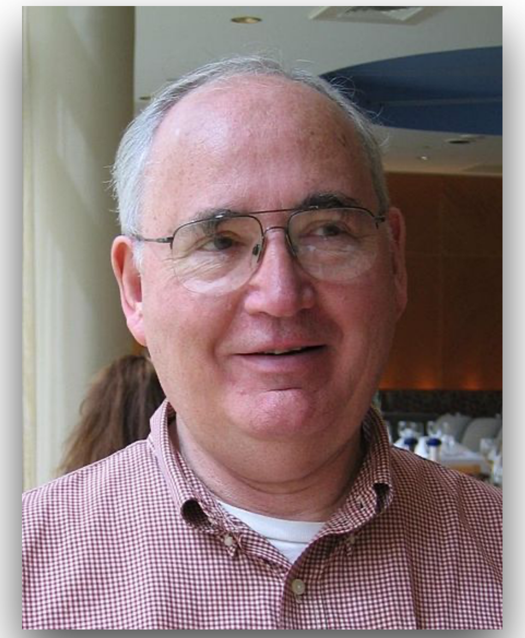
William Cook*
Department of Computer Science
Box 1910 Brown University

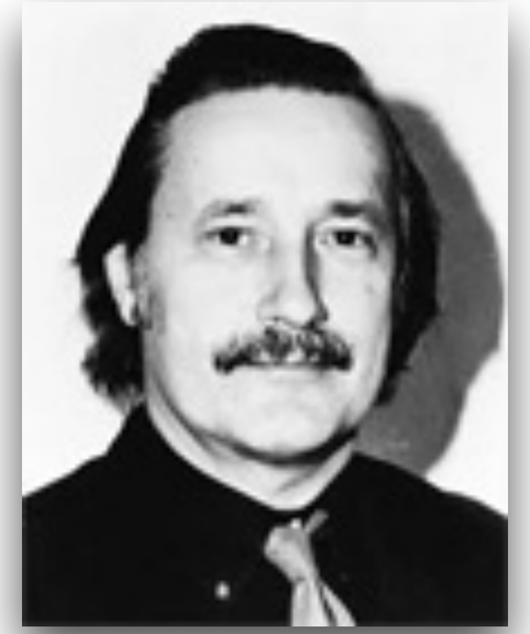
Jens Palsberg
Computer Science Department
Aarhus University

This paper presents a denotational model of inheritance. The model is based on an intuitive motivation of the purpose of inheritance. The correctness of the model is demonstrated by proving it equivalent to an operational semantics of inheritance based upon the method-lookup algorithm of object-oriented languages. ...

OOPSLA '89: Conference proceedings on Object-oriented programming systems, languages and applications

Denotational semantics
– Scott–Strachey style





Denotational semantics

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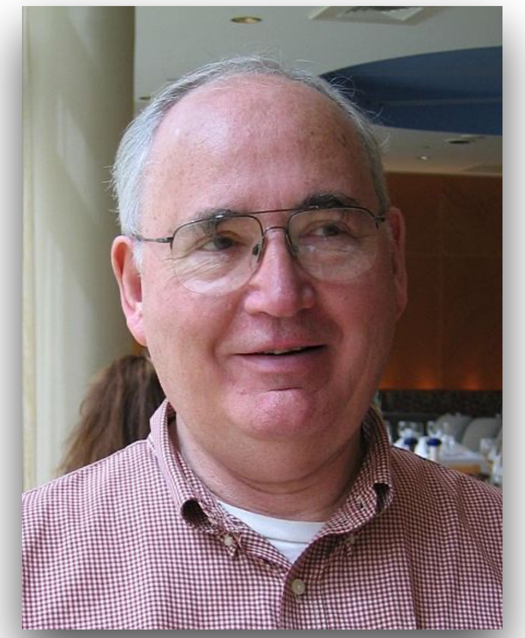

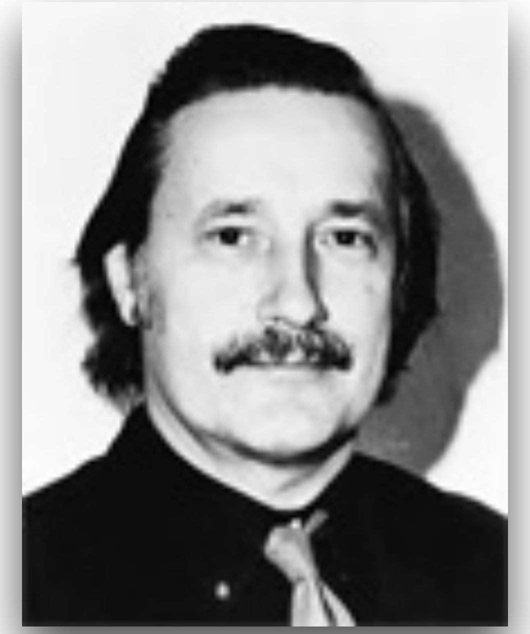

Types of denotations are (Scott-)domains

- ***pointed cpos*** (e.g, ω-complete, directed-complete, continuous lattices)
- ***recursively defined*** – without guards, up to isomorphism

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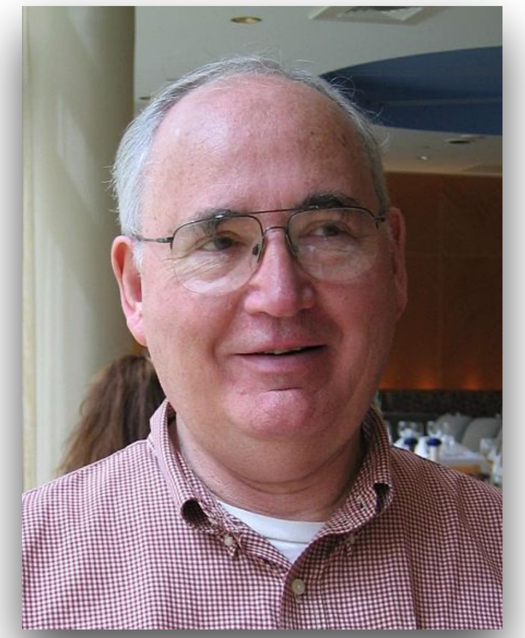

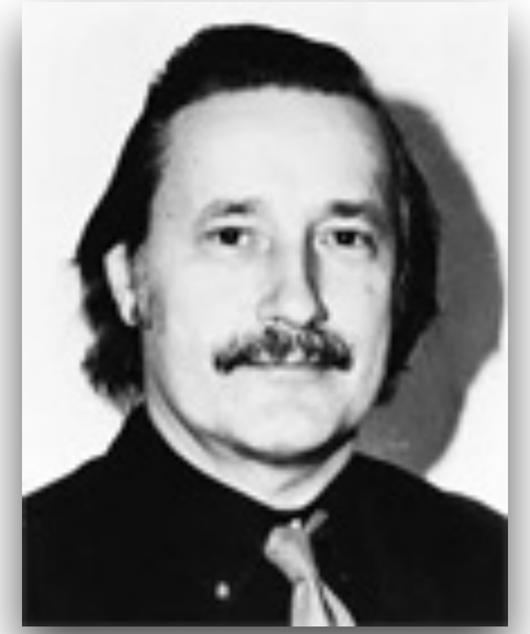

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Denotations are defined in typed λ-notation

- functions on domains are ***continuous maps***
- endofunctions on domains have least ***fixed points***

Models of the untyped λ-calculus
– based on Scott's domain D_∞

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Some mathematical presentations:

- *Dana Scott* (1970,1972): continuous lattices, D_∞
- *Joseph Stoy* (1977): universal domain $\mathscr{P}\omega$
- *Samson Abramsky and Achim Jung* (1994): (pre)domain theory
- *John Reynolds* (2009): *Theories of Programming Languages*, cpos, D_∞

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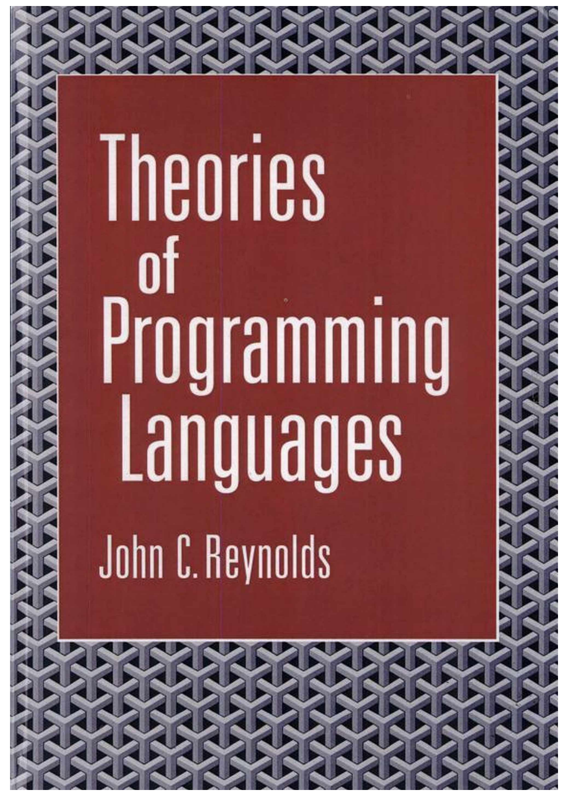
Some formalizations:

- *Bernhard Reus* (1994): using *Extended Calculus of Constructions*, in *Lego*
- *Tom de Jong* (2021): using *Univalent Type Theory* (TypeTopology), in *Agda*

Reynolds: Theories of Programming Languages

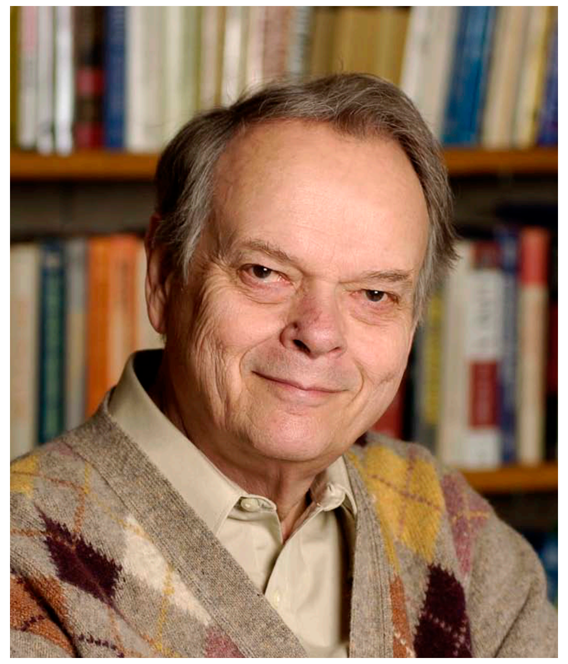
– denotational semantics of the untyped λ-calculus



isomorphism

continuous maps

$$D_\infty \underset{\psi}{\overset{\phi}{\rightleftarrows}} [D_\infty \to D_\infty]$$



$$\begin{aligned}
[\![-]\!] &\in exp \to [(var \to D_\infty) \to D_\infty] \\
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[\![e\, e']\!]\,\eta &= \phi\,([\![e]\!]\,\eta)\,([\![e']\!]\,\eta)
\end{aligned}$$

Copied from www.cs.yale.edu/homes/hudak/CS430F07/LectureSlides/Reynolds-ch10.pdf

Reynolds: Theories of Programming Languages

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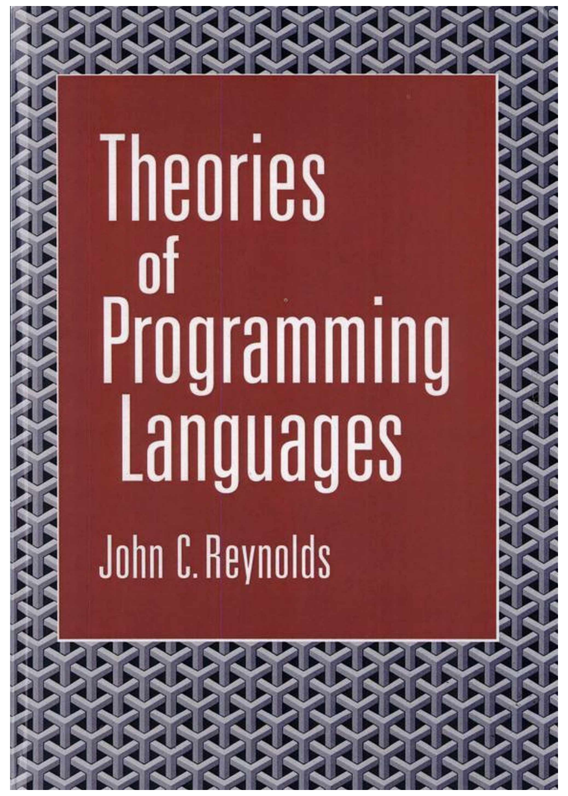

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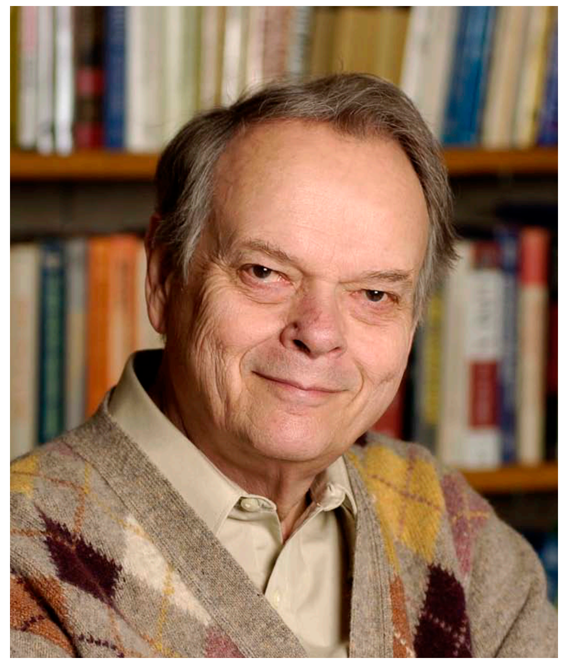

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Agda formalization

– using TypeTopology/DomainTheory (Tom de Jong)

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$$\mathsf{abs} : \langle\, \mathcal{D}_\infty \Longrightarrow^{\text{dcpo}} \mathcal{D}_\infty \,\rangle \to \langle\, \mathcal{D}_\infty \,\rangle$$

$$\mathsf{app} : \langle\, \mathcal{D}_\infty \,\rangle \to \langle\, \mathcal{D}_\infty \,\rangle \to \langle\, \mathcal{D}_\infty \,\rangle$$

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$$\mathsf{abs} : \langle\ \mathcal{D}_\infty \Longrightarrow^{\mathrm{dcpo}} \mathcal{D}_\infty\ \rangle \to \langle\ \mathcal{D}_\infty\ \rangle$$
$$\mathsf{abs} = [\ \mathcal{D}_\infty \Longrightarrow^{\mathrm{dcpo}} \mathcal{D}_\infty\ ,\ \mathcal{D}_\infty\ ]\langle\ \pi\text{-}\mathsf{exp}_\infty\text{'}\ \rangle$$

$$\mathsf{app} : \langle\ \mathcal{D}_\infty\ \rangle \to \langle\ \mathcal{D}_\infty\ \rangle \to \langle\ \mathcal{D}_\infty\ \rangle$$
$$\mathsf{app} = \mathsf{underlying\text{-}function}\ \mathcal{D}_\infty\ \mathcal{D}_\infty \circ [\ \mathcal{D}_\infty\ ,\ \mathcal{D}_\infty \Longrightarrow^{\mathrm{dcpo}} \mathcal{D}_\infty\ ]\langle\ \varepsilon\text{-}\mathsf{exp}_\infty\text{'}\ \rangle$$

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a continuous function is a ***pair:***
- an *underlying* function and
- a *proof* of its continuity

Agda formalization
– using TypeTopology/DomainTheory (Tom de Jong)

$[\![\_]\!] : \mathsf{Exp} \to \mathsf{Env} \to \langle\, \mathcal{D}_\infty \,\rangle$
$ƛ\text{-is-continuous} : \forall\, e\ \rho\ v \to \text{is-continuous}\ \mathcal{D}_\infty\ \mathcal{D}_\infty\ (\lambda\, x \to [\![\, e \,]\!]\ (\rho\ [\, x \,/\, v \,]))$

$[\![\ \mathsf{var}\ v\ ]\!]\ \rho = \rho\ v$
$[\![\ ƛ\ v \cdot e\ ]\!]\ \rho = \mathsf{abs}\ (\ (\lambda\, x \to [\![\, e \,]\!]\ (\rho\ [\, x \,/\, v \,]))\ ,\ ƛ\text{-is-continuous}\ e\ \rho\ v\ )$
$[\![\ e_1 \cdot e_2\ ]\!]\ \rho = \mathsf{app}\ (\ [\![\ e_1\ ]\!]\ \rho\ )\ (\ [\![\ e_2\ ]\!]\ \rho\ )$

Agda formalization

– using TypeTopology/DomainTheory (Tom de Jong)

```
⟦_⟧ : Exp → Env → ⟨ 𝒟∞ ⟩
ƛ-is-continuous : ∀ e ρ v → is-continuous 𝒟∞ 𝒟∞ (λ x → ⟦ e ⟧ (ρ [ x / v ]))

⟦ var v ⟧ ρ = ρ v
⟦ ƛ v · e ⟧ ρ = abs ( (λ x → ⟦ e ⟧ (ρ [ x / v ])) , ƛ-is-continuous e ρ v)
⟦ e₁ · e₂ ⟧ ρ = app ( ⟦ e₁ ⟧ ρ ) ( ⟦ e₂ ⟧ ρ )

ƛ-is-continuous e ρ v = {!  !}
```

Lightweight Agda formalization

– modules

Abstract syntax grammar

- inductive ***datatype definitions***

'Domain' definitions

- ***postulated isomorphisms*** between ***type names*** and ***type terms***

Semantic functions

- functions defined ***inductively*** in ***λ-notation***

Auxiliary definitions

Lightweight Agda formalization

– abstract syntax

```
data Exp : Set where
  var_ : Var → Exp
  lam  : Var → Exp → Exp
  app  : Exp → Exp → Exp
```

Lightweight Agda formalization

– a 'domain'

```
open import Function
  using (Inverse; _↔_) public
open Inverse {{ ... }}
  using (to; from) public

postulate
  D∞ : Set
postulate
  instance iso : D∞ ↔ (D∞ → D∞)
```

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bijection

Agda functions

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bijection

Agda functions

Lightweight Agda formalization

– semantic function

$$\mathsf{Env} = \mathsf{Var} \to D_\infty$$

$$[\![\_]\!] : \mathsf{Exp} \to \mathsf{Env} \to D_\infty$$

$$\begin{aligned}
[\![\ \mathsf{var}\ v\ ]\!]\ \rho &= \rho\ v \\
[\![\ \mathsf{lam}\ v\ e\ ]\!]\ \rho &= \boxed{\mathsf{from}}\ (\ \lambda\ d \to [\![\ e\ ]\!]\ (\rho\ [\ d\ /\ v\ ])\ ) \\
[\![\ \mathsf{app}\ e_1\ e_2\ ]\!]\ \rho &= \boxed{\mathsf{to}}\ (\ [\![\ e_1\ ]\!]\ \rho\ )\ (\ [\![\ e_2\ ]\!]\ \rho\ )
\end{aligned}$$

Lightweight Agda formalization

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Lightweight Agda formalization

– testing denotations

$$\text{check-convergence} : (\lambda x_1 \,.\, x_{42})((\lambda x_0 \,.\, x_0 x_0)(\lambda x_0 \,.\, x_0 x_0)) \equiv x_{42}$$

Lightweight Agda formalization
– testing denotations

check-convergence : $(\lambda x_1 \,.\, x_{42})((\lambda x_0 \,.\, x_0 x_0)(\lambda x_0 \,.\, x_0 x_0)) \equiv x_{42}$

```
check-convergence :
  ⟦ app (lam (x 1) (var x 42))
        (app (lam (x 0) (app (var x 0) (var x 0)))
             (lam (x 0) (app (var x 0) (var x 0)))) ⟧
  ≡ ⟦ var x 42 ⟧
check-convergence = refl
```

Lightweight Agda formalization

– testing denotations

```
to-from-elim : ∀ {f} → to (from f) ≡ f
to-from-elim = inverseˡ iso refl
{-# REWRITE to-from-elim #-}

check-convergence : (λx₁ . x₄₂)((λx₀ . x₀x₀)(λx₀ . x₀x₀)) ≡ x₄₂
  ⟦ app (lam (x 1) (var x 42))
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```
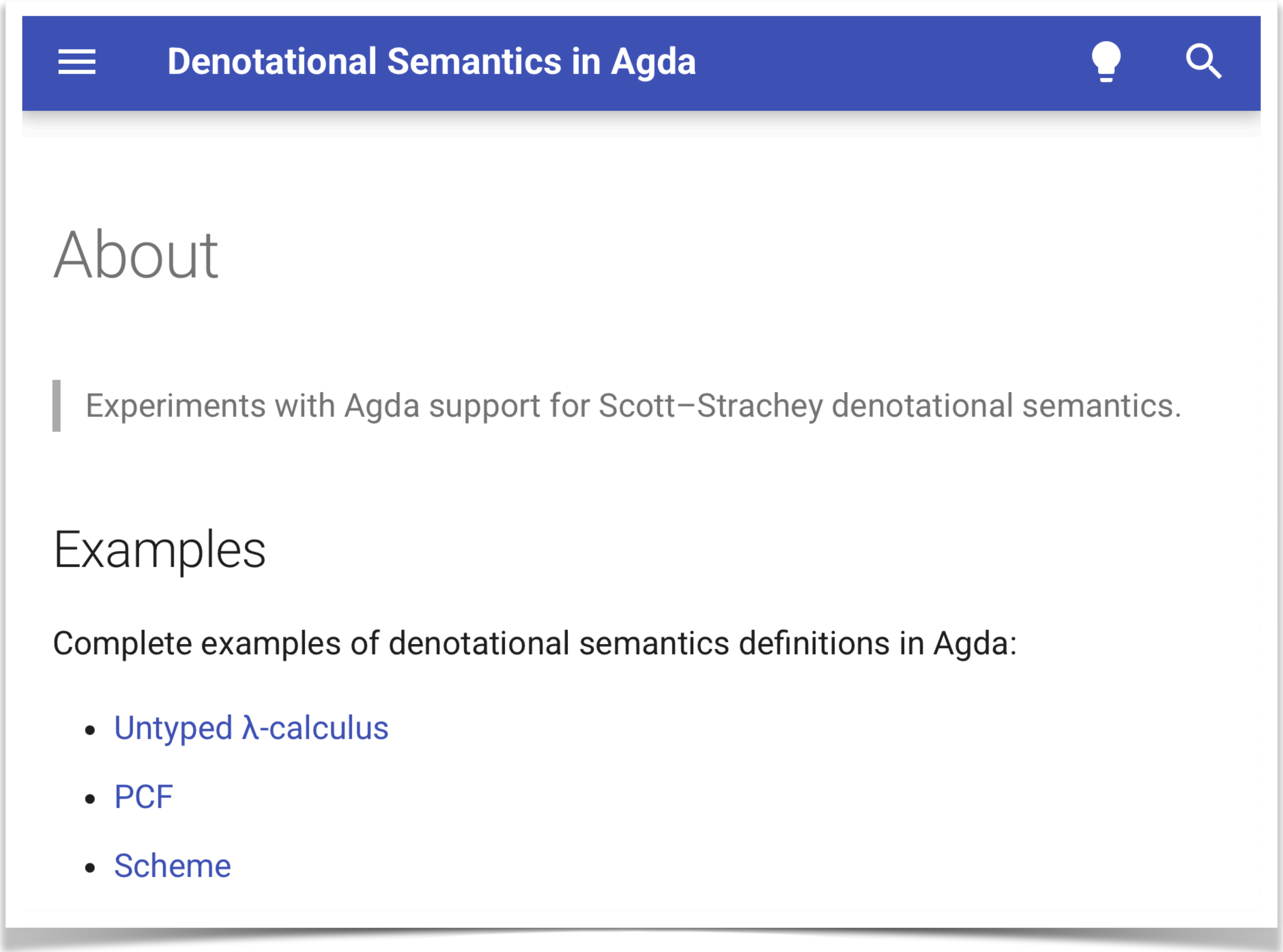

– potentially unsafe!

Other examples: PCF, Scheme

– pdmosses.github.io/xds-agda/



Denotational Semantics in Agda

About

> Experiments with Agda support for Scott–Strachey denotational semantics.

Examples

Complete examples of denotational semantics definitions in Agda:

- Untyped λ-calculus
- PCF
- Scheme

Safe **lightweight Agda formalization?**

– future work

Implement SDT (Synthetic Domain Theory)

- use ***plain*** Agda
- embed Agda types as ***predomains***
- assume only properties ***consistent*** with MLTT
- make functions ***implicitly*** continuous
- allow ***unrestricted*** recursive domain definitions
- …