Choice principles and a cotopological modality in HoTT

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Anel and Barton [1] define a family of generalized forms of the axiom of countable choice, and discuss their relationship to Postnikov completeness and the hypercompletion operation in ∞-toposes. This talk will discuss the translation of their work into HoTT and a formalization in Cubical Agda. In particular, we will present a construction of the hypercompletion operation in HoTT under the assumption that any one of those forms of the axiom of countable choice holds.

Working in intensional Martin-Löf type theory with a univalent universe, we say that a type is (-2)-truncated if it is equal to the unit type. A type is called $(n+1)$-truncated $(-2 \leq n < \infty)$ when its identity types are n-truncated. For any $n < \infty$, there is an operation $\lambda X.\|X\|_n$ sending a type to its *n-truncation* – which is a *universal* n-truncated type with a map $|\cdot|_n : X \to \|X\|_n$.

We say that a modality is any operation $\bigcirc : \mathcal{U} \to \mathcal{U}$ together with a *unit* $\eta : (X : \mathcal{U}) \to X \to \bigcirc X$, which satisfy certain conditions (see [4] for the definitive account). If η_X is an equivalence, we say that X is a modal type for the modality $(\bigcirc, \eta)$. The n-truncation operation, together with the universal map, is a modality, its modal types are precisely the n-truncated types.

A type X is called n-connected if $\|X\|_n$ is equal to the unit type. A type Y is n-truncated if and only if, for all n-connected types X, the canonical map $Y \to (X \to Y)$ is an equivalence. We say that a type is ∞-connected if $\|X\|_n$ is equal to the unit type *for all* n. By analogy with the finite case, we say that a type Y is ∞-truncated if, for all ∞-connected types X, the canonical map $Y \to (X \to Y)$ is an equivalence.

An important construction in ∞-topos theory is the *hypercompletion* operation, which sends each object in an ∞-topos to an ∞-truncated reflection of that object [2]. Since HoTT has semantics in ∞-toposes [5], it is desirable to construct such an operation in the type theory. In particular, this ought to be a modality whose modal types are precisely the ∞-truncated types. However, the authors of [4] remark that it is unknown how to construct such a modality in HoTT without additional assumptions. As far as I am aware, little subsequent progress has been made on the problem.

That said, we clearly can construct this modality under particular assumptions. In particular – if we assume that each type is already ∞-truncated, then the hypercompletion agrees with the trivial modality. However, as we're going to see, this is not the only condition which allows us to construct this modality.

Postnikov towers

A formal Postnikov tower is a family of types $A : \mathbb{N} \to \mathcal{U}$, with a family of maps $a : (n : \mathbb{N}) \to A_{n+1} \to A_n$, such that for each $n : \mathbb{N}$, A_n is n-truncated and a_n has n-connected fibers. For every type X there is a canonical Postnikov tower with $A_n := \|X\|_n$ and $a_n := |\cdot|_n$. For any tower (Postnikov or otherwise) we can define its limit:

$$\lim(A, a) := (x : (n : \mathbb{N}) \to A_n) \times ((n : \mathbb{N}) \to a_n(x_{n+1}) = x_n)$$

There is a canonical morphism from a type X into the limit of its canonical Postnikov tower. Postnikov convergence is the statement that these canonical morphisms are always equivalences. Meanwhile, if (A, a) is a Postnikov tower, there is a canonical family of maps $\epsilon_n : \| \lim(A, a)\|_n \to A_n$, and Postnikov effectiveness is the statement that for every Postnikov tower, each ϵ_n is an equivalence. The conjunction of Postnikov convergence and Postnikov effectiveness is called Postnikov completeness.

Postnikov convergence implies that every type is ∞-truncated – so, again, the hypercompletion modality agrees with the trivial modality if Postnikov convergence holds. Postnikov effectiveness also allows us to construct a hypercompletion modality and in this case (so long as Postnikov convergence does not also hold) it is non-trivial – the operation sends a type to the limit of its canonical Postnikov tower:

$$\lambda X. \lim(\|X\|_n, |\cdot|_n)$$

So far so good, but actually we can get away with a simpler assumption, one which does not refer to Postnikov towers. Anel and Barton [1] define countable choice of dimension $\leq d$ for a fixed ∞-topos to be the principle stating that if $X_1, X_2, \ldots$ form a family of $d + k$-connected objects, then their product $\Pi_{\mathbb{N}} X_n$ is k-connected. They show that each one of these principles implies (an external version of) Postnikov effectiveness for the topos. They observe that their definitions and most of their arguments can be reformulated in HoTT. For a formalization in Cubical Agda see the repository: [3]. All of this allows us to conclude in HoTT that for any d, countable choice of dimension $\leq d$ allows us to construct a hypercompletion modality.

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