Synthetic-Inductive Category Theory

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One key insight emerging from the development of Martin-Löf Type Theory is that *types are synthetic groupoids*. That is, the identity types of a given type A possess the structure of a groupoid—the terms are the objects, refl_t is the identity morphism on $t\colon A$, the invertibility of identity terms encodes the fact that every morphism is an isomorphism, and so on. The "lower-dimensional" version of this observation was made in the 1990s, most famously in Hofmann and Streicher's *groupoid model* [3]. In the groupoid model, the types are synthetic 1-groupoids: the model's refutation of the *uniqueness of identity proofs* corresponds to the fact that there may be multiple distinct morphisms between any two objects in a groupoid, but the identity structure becomes propositional beyond that point. Using type theory as a synthetic language for *higher* groupoids, indeed ∞-*groupoids*, was a key impetus behind the development of *homotopy type theory* [10]. In homotopy type theory, the types are natively higher groupoids [11], with the identity types possessing (potentially) nontrivial identity type structure of their own, and those identity types having interesting identity types, and so on.

These developments led naturally to the question: what about synthetic *categories*? That is, can the symmetry of identity types be dropped (yielding "hom-types"), so that our types are instead natively endowed with the structure of synthetic *categories*? This question proves to be more challenging, but the outlook of such **directed type theory** is quite hopeful. Approaches to directed type theory also come in higher- and lower-dimensional flavours. In recent years, there has been a significant amount of interest in the *simplicial* approach to synthetic ∞-category theory pioneered by Riehl and Shulman[9], with important pieces of ∞-category theory being performed in the synthetic style [12, 1, 2] and an experimental proof assistant RZK being developed to allow for computer formalisation [4]. The lower-dimensional track—a "directed analogue" of Hofmann-Streicher—is less-developed: though the appropriate definition of the *category model* and its hom-types are known [8], there has yet to be a consensus on how to arrange the category model's *polarity calculus* in such a way to make **synthetic 1-category theory** possible. In recent work [7, 6], I[1] propose a resolution to these issues (combining aspects of the type theories of North [8] and Licata-Harper [5], plus further innovations), and show how the category model interprets a directed type theory capable of synthetic 1-category theory.

The purpose of this talk is to expound this directed type theory, and show the style of synthetic category theory it permits. Following the HoTT Book [10], we adopt an informal style more suited to mathematical practice. In this style of directed type theory, a modal typing discipline is adopted to track the *variances* of terms: $t\colon A$ means that t is *covariant*, whereas $t\colon A^-$ means that t is *contravariant*. Viewing types as synthetic categories, this is the same as saying that A^- is the *opposite category* of A. Accordingly, we type our hom-type formation rule as follows.

$$x\colon A^-, y\colon A \vdash \hom(x, y)\ \text{type}$$

These polarities have a *substructural* character: for *closed* terms we are able to freely coerce between A^- and A (for every closed term $t\colon A^-$, there is a closed term $-t\colon A$ and *vice-versa*)

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but for *open* terms we have no such operation. Therefore, the reflexivity term—the identity morphisms in our synthetic category theory—are stated only for closed terms.

$$\frac{t\colon A^-}{\mathsf{refl}_t\colon \hom(t,-t)}$$

The purpose of these restrictions is to prevent the elimination principle for hom-types—the *directed J-rule*—from being able to prove symmetry. We can prove by metatheoretic argument that our directed J-rule,

$$\frac{\begin{array}{c} t\colon A^- \\ y\colon A, u\colon \hom(t,y) \vdash M(y,u)\ \text{type} \\ m\colon M(-t,\mathsf{refl}_t) \end{array}}{y\colon A, u\colon \hom(t,y) \vdash J_{t,M}(m)\colon M(y,u)}$$

cannot prove symmetry: it is validated by the category model, but the category model refutes symmetry (not every category is a groupoid). Using this principle of directed path induction, we can easily construct the composition of hom-terms [2] and prove it satisfies the category laws.

When we carry out synthetic category theory in this setting, it permits a novel style of category theory, which we call **inductive category theory**. In this framework, the *universal mapping properties* of traditional category theory are instead reframed as *principles of induction*. For instance, we make the following definition: given terms $s, t\colon A$, we say that a term $s\times t\colon A^-$ and terms $\pi_1\colon \hom(s\times t, s)$ and $\pi_2\colon \hom(s\times t, t)$ constitute a *product* of s and t if it satisfies the following principle of induction.

$$\frac{\begin{array}{c} z\colon A^-, u\colon \hom(z,s), v\colon \hom(z,t) \vdash M(z,u,v)\ \text{type} \\ m\colon M(s\times t, \pi_1, \pi_2) \end{array}}{z\colon A^-, u\colon \hom(z,s), v\colon \hom(z,t) \vdash \mathsf{elim}(m)\colon M(z,u,v)}$$

From here, we can recover the usual universal mapping property. For instance, given a *cone* $k\colon A^-$, $f\colon \hom(k,s)$, $g\colon \hom(k,t)$, we can obtain the hom-term $\langle f,g\rangle\colon \hom(k,-(s\times t))$ by induction: just set $\langle \pi_1,\pi_2\rangle$ equal to $\mathsf{refl}_{s\times t}$ and we're done.

In this talk, we'll explore this peculiar method of phrasing category-theoretic concepts, showing that it can encompass more complex notions like exponentials and adjoints, and speculate on what a fully-worked out category theory in this style might look like.

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