Predicative Stone Duality in Univalent Foundations

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Summary. Stone duality for spectral locales [8] states that the category of spectral locales and spectral maps is dually equivalent to the category of distributive lattices. In this work, we construct this equivalence in the setting of constructive and predicative univalent foundations (UF), by working with the category of large, locally small, and small-complete spectral locales. This extends the authors' previous work [1] in which a predicatively well-behaved definition of the notion of spectral locale was proposed in the setting of UF. In showing that this notion of spectrality enjoys Stone duality, we formally establish that it captures the intended category of spectral locales.

The results presented here will soon appear in the second named author's PhD thesis [9].

Background on Stone duality. In an impredicative setting, a *spectral locale* [7, p. 63] is defined[1] as a locale in which the compact opens form a base closed under finite meets. A continuous map of spectral locales is called spectral if it reflects compact opens.

A *distributive lattice* (also called *bounded distributive lattice* in the literature) is a lattice with finite meets and finite joins in which the meets distribute over the joins (and vice versa). Given a spectral locale X, the collection of its compact opens (denoted $\mathsf{K}(X)$) forms a distributive lattice as the compact opens are always closed under finite joins, and also closed under finite meets in spectral locales. Conversely, given a distributive lattice L, the collection of ideals of L (denoted $\mathsf{Idl}(L)$) forms a spectral frame. The locale defined by this frame is called the *spectrum of* L and is denoted $\mathsf{spec}(L)$. Stone duality for spectral locales amounts to the fact that these two maps form a categorical equivalence when extended to functors.

Foundations. We work in the context of intensional MLTT extended with univalent features. We assume the existence of Σ and Π types as well as the inductive types of $\mathbf{0}$, $\mathbf{1}$, the natural numbers, and lists. Furthermore, we assume univalence and propositional truncations. Instead of assuming the univalence axiom globally, we assume specific universes in consideration to be univalent. We define the type of families (with index type living in universe $\mathcal{W}$) on a given type A as $\mathsf{Fam}_{\mathcal{W}}(A) :\equiv \Sigma_{(I\,:\,\mathcal{W})} I \to A$. We often use the abbreviation $(x_i)_{i:I}$ for a family $x : I \to A$.

We work explicitly with universes. We denote the ground universe by $\mathcal{U}_0$, the successor of a universe $\mathcal{U}$ by $\mathcal{U}^+$, and the least upper bound of two universes $\mathcal{U}$ and $\mathcal{V}$ by $\mathcal{U} \sqcup \mathcal{V}$.

A type $X : \mathcal{U}$ is called *$\mathcal{V}$-small* if it is equivalent to a specified type in universe $\mathcal{V}$ i.e. $\Sigma_{(Y\,:\,\mathcal{V})} X \simeq Y$, and is called *locally $\mathcal{V}$-small* if the identity type $x =_X y$ is $\mathcal{V}$-small, for every pair of inhabitants $x, y : X$. A universe $\mathcal{U}$ is said to be *univalent* if the canonical map $\mathsf{idtoeqv} : X =_{\mathcal{U}} Y \to X \simeq Y$ is an equivalence, and the *univalence axiom* says that all universes are univalent. The univalence axiom implies that the type expressing that X is $\mathcal{V}$-small is a proposition, for every pair of universes $\mathcal{U}$ and $\mathcal{V}$, and every type $X : \mathcal{U}$. In fact, this can be extended to a logical equivalence [4]: if X being $\mathcal{V}$-small is a proposition for every pair of universes $\mathcal{U}$ and $\mathcal{V}$ and every type $X : \mathcal{U}$, then the univalence axiom holds.

[1] This notion is also called a *coherent locale* in the literature.

Locale theory with explicit universes in UF. We fix a base universe $\mathcal{U}$ and we call $\mathcal{U}$-small types simply *small* in this context. A lattice is called (1) *large* if its carrier set lives in universe $\mathcal{U}^+$ and *small* otherwise, (2) *locally small* if its order has small truth values, and (3) *small-complete* if it has joins of small families. It is well known that complete, small lattices cannot be constructed in a predicative foundational setting without using a form of propositional resizing. This was first shown by Curi [2, 3] for CZF, and then by de Jong and Escardó [5, 4] for predicative UF. Due to this no-go theorem, our investigation of locale theory focuses on the category of large, locally small, and small-complete frames.

Spectral locales in a predicative setting. A family $(B_i)_{i:I}$ of opens in some locale X is said to form a *weak base* for X if, for every open U, there is an unspecified subfamily $(B_{i_j})_{j:J}$ such that $U = \bigvee_{j:J} B_{i_j}$. A locale X is called *spectral* if it satisfies the following four conditions: (SP1) it is compact (i.e. the empty meet is compact), (SP2) the meet of two compact opens is compact, (SP3) the family $\mathsf{K}(X) \hookrightarrow \mathcal{O}(X)$ forms a weak base, and (SP4) the type $\mathsf{K}(X)$ is small.

Compared to the standard definition, the most salient difference here is the stipulation that the type $\mathsf{K}(X)$ be small. The justification for having requirement in the definition is twofold: (1) many fundamental constructions of locale theory (e.g. Heyting implications, open nuclei, right adjoints of frame homomorphisms) do not seem to be available in the absence of a small (weak) base, and (2) a natural desideratum for the definition of spectrality is that it satisfy Stone duality, and it can be shown, as we will explain soon, that *every* locale homeomorphic to the spectrum of a (small) distributive lattice satisfies the above definition. We denote by $\mathbf{Spec}_{\mathcal{U}}$ the category of large, locally small, and small-complete spectral locales over universe $\mathcal{U}$.

Propositionality of being spectral. In UF, it is natural to expect that being spectral is a property and not structure. If we assume the base universe to be univalent, we can show that this is indeed a proposition, since this is what we need to conclude that Condition (SP4) is a proposition. As explained previously, however, the propositionality of being small is logically equivalent to univalence, which is to say that the use of univalence here seems to be essential.

Spectrum construction with explicit universes. A small distributive lattice (over base universe $\mathcal{U}$) is a small set $L : \mathcal{U}$ equipped with operations $\text{-}\wedge\text{-} : L \to L \to L$ and $\text{-}\vee\text{-} : L \to L \to L$, which are associative and commutative, have unit elements, and satisfy the absorption laws i.e. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$, for every $x, y : L$. We denote by $\mathbf{DLat}_{\mathcal{U}}$ the category of small distributive lattices over base universe $\mathcal{U}$. By a *small ideal* of L, we mean a $\mathcal{U}$-valued subset $S : L \to \Omega_{\mathcal{U}}$ that is inhabited, downward closed, and closed under binary joins. We show that the small ideals of a lattice L form a large, locally small, and small-complete locale satisfying the above definition of spectrality. We then extend this to a functor $\mathsf{spec} : \mathbf{DLat}_{\mathcal{U}} \to \mathbf{Spec}_{\mathcal{U}}$.

Predicative Stone duality. We show that the type $\mathsf{K}(X)$ is a small distributive lattice, for every large, locally small, and small-complete spectral locale X. This exploits Condition (SP4), stipulating the smallness of $\mathsf{K}(X)$. We then extend this to a functor $\mathsf{K} : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$, and show that it forms a categorical equivalence when paired with $\mathsf{spec} : \mathbf{DLat}_{\mathcal{U}} \to \mathbf{Spec}_{\mathcal{U}}$.

Formalization. Most of the work we present has been formalized using the Agda proof assistant, as part of the TypeTopology library [6]. The object part of the categorical equivalence has been completely formalized, and the extension of this to the full categorical equivalence is work in progress at the time of writing.

References

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