Cocompleteness in simplicial homotopy type theory

Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz

1 Introduction

Simplicial type theory (STT) was introduced by Riehl and Shulman [8] to give a *synthetic* account of ∞-category theory[1] based on homotopy type theory. Specifically, STT extends HoTT with a handful of axioms to tune the theory to a particular collection of models (those based on the ∞-topoi $\mathcal{E}^{\Delta^{\mathsf{op}}}$) into which the ∞-category of small (internal) ∞-categories fully-faithfully embeds [7]. The most important axiom postulates a type $\mathbb{I}$ which behaves like a *directed interval*:

Axiom 1. *We assume an h-set $\mathbb{I} : \mathcal{U}$ such that $\mathbb{I}$ is a totally-ordered bounded distributive lattice.*

We may then use $\mathbb{I}$ to associate to any $x, y : X$ the type of *synthetic morphisms* $\hom(x, y)$:

$$\hom(x, y) = \sum_{f:\mathbb{I}\to X} f(0) = x \times f(1) = y$$

Not every type admits a composition operation, so we isolate the data of a composition of two morphisms using $\Delta^2 = \sum_{i,j:\mathbb{I}} i \geq j$. In particular, $\alpha : \Delta^2 \to X$ intuitively classifies a pair of composable morphisms in X—$\alpha(1, -)$ and $\alpha(-, 0)$—together with their composite—$\lambda i.\, \alpha(i, i)$.

Definition 1.1. A pre-category X is a type where the restriction $X^{\Delta^2} \to X^{\mathbb{I}} \times_X X^{\mathbb{I}}$ is an equivalence. An inverse to this restriction is denoted $\circ$ and composes morphisms in X.

While a pre-category has composition operation, to correctly model ∞-categories we further restrain X such that $x =_X y$ coincides with synthetic isomorphisms $x \cong y$:

Definition 1.2. A pre-category X is a category if $\mathsf{isEquiv}\big(\lambda x.(x, x, \mathsf{id}) : X \to \sum_{x,y} x \cong y\big)$.

Definition 1.3. A groupoid is a category where all morphisms are isomorphisms.

On top of this handful of basic definitions, Riehl and Shulman [8] developed the theory of functors, natural transformations, and adjoints within STT and since then authors [1–3, 5, 6, 10, 11] have worked to reproduce or extend ∞-category theory within STT. In particular, the recent work by Gratzer et al. [5, 6] extends STT with a collection of *modalities* [4] to construct the category of groupoids $\mathcal{S}$ along with the theory of presheaf categories.

We extend this line of research on modal STT with a series of results in showing that cocompleteness may be reduced to the existence of various simpler colimits. We are then able to provide an entirely synthetic account of (generalized) homology and cohomology theories.

Notation 1.4. While the details of the modal STT are not relevant, we note that it generalizes Shulman [9] and, in particular, includes the ability to quantify over only the objects of categories i.e., to treat such elements non-functorially. These non-functorial variables are annotated by $\flat$.

2 Colimits and cocompleteness

Our focus is to analyze conditions equivalent to the following:

Definition 2.1. A category C is *cocomplete* if $C \to C^I$ is a right adjoint for all $I :_\flat \mathcal{U}_0$.

[1] By ∞-category theory we specifically mean $(\infty, 1)$-category theory.

Our main results offer simpler alternative conditions for a category to be cocomplete.

Theorem 2.2. *A category C is cocomplete if any of the following hold:*

1. *C has pushouts as well as colimits indexed by groupoids* and *crisp excluded middle holds (for every h-proposition $\phi :_\flat \mathcal{U}$ either ϕ or $\neg\phi$ holds).*
2. *C has finite coproducts and all sifted colimits*
3. *C has all finite colimits and all filtered colimits.*

The latter two conditions are both necessary and sufficient.

In the above, filtered and sifted colimits are defined using the notion of cofinal maps as introduced in STT by Gratzer, Weinberger, and Buchholtz [6] and closely follow the standard definitions from ∞-category theory: A category C is sifted if $C \to C^n$ is right cofinal for all $n : \mathsf{Nat}$ and filtered if $C \to C^K$ is right cofinal for all finite complexes K. In fact, in *ibid.* it is shown that the category of $\mathcal{S}$-valued presheaves on C is the free cocompletion.

3 Spectra and (co)homology

We apply the above results to the category of *spectra* Sp. If (∞-)groupoids replace the category of sets in ∞-category theory, spectra take the place of abelian groups. We are now able to show that Sp is stable (Lemma 3.3) and use this to construct homology theories satisfying the Eilenberg–Steenrod axioms.

Definition 3.1. Sp is given by the limit (computed as in Book HoTT) $\varprojlim(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \dots)$.

Lemma 3.2. Sp *has all filtered colimits and all limits.*

Lemma 3.3. Sp *is finitely (co)complete,* $\mathbf{0}_{\mathsf{Sp}} \cong \mathbf{1}_{\mathsf{Sp}}$*, and pushouts and pullbacks coincide.*

Corollary 3.4. Sp *is cocomplete.*

Proof. Apply Theorem 2.2 to Lemmas 3.2 and 3.3. □

A fundamental example of a spectrum is $H\mathbb{Z}$, the Eilenberg–MacLane spectrum given by the sequence of Eilenberg–MacLane spaces $(K(\mathbb{Z},0), K(\mathbb{Z},1), \dots)$. By Gratzer, Weinberger, and Buchholtz [6], there is an equivalence between cocontinuous maps $\mathcal{S} \to \mathsf{Sp}$ and elements of spectra and we write $H : \mathcal{S} \to \mathsf{Sp}$ for the cocontinuous functor induced by $H\mathbb{Z}$. We further write $\pi_i : \mathsf{Sp} \to \mathsf{Ab}$ for the functor sending X to $\pi_0(\mathsf{proj}_i(X))$.

Lemma 3.5. *If $X \to Y$ and $Z = Y \sqcup_X \mathbf{1}$ then there is a long exact sequence of abelian groups:*

$$\pi_i X \longrightarrow \pi_i Y \longrightarrow \pi_i Z \longrightarrow \pi_{i-1} X \longrightarrow \dots$$

Theorem 3.6. *The functors $\pi_i \circ H : \mathcal{S} \to \mathsf{Ab}$ satisfies the Eilenberg–Steenrod axioms.*

Proof. Of the Eilenberg–Steenrod axioms (homotopy invariance, excision, dimension, and additivity), homotopy invariance and dimension follow more-or-less by definition. Excision states that cofibers in $\mathcal{S}$ are sent to long exact sequences. This is a corollary of the cocontinuity of H alongside Lemma 3.5. The additivity axiom states that that coproducts are preserved by H_i. For finite coproducts, Lemma 3.3 implies that finite coproducts agree with finite products and $\pi_i : \mathsf{Sp} \to \mathsf{Ab}$ sends finite products to direct sums by calculations. For the general case, one decomposes $\coprod_{i:I} X_i$ for a discrete set I into the filtered colimit of finite colimits $\varinjlim_{(n,f):\sum_{n:\mathbb{N}} I^n} \coprod_{k \le n} X_{f(i)}$ to reduce to the finite case. □

References

[1] César Bardomiano Martínez. *Exponentiable functors between synthetic ∞-categories*. 2024. arXiv: 2407.18072.

[2] César Bardomiano Martínez. *Limits and colimits of synthetic ∞-categories*. 2022. arXiv: 2202.12386.

[3] Ulrik Buchholtz and Jonathan Weinberger. "Synthetic fibered $(\infty, 1)$-category theory". In: *Higher Structures* 7 (1 2023), pp. 74–165. DOI: 10.21136/HS.2023.04.

[4] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. "Multimodal Dependent Type Theory". In: *Logical Methods in Computer Science* Volume 17, Issue 3 (July 2021). DOI: 10.46298/lmcs-17(3:11)2021.

[5] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *Directed univalence in simplicial homotopy type theory*. 2024. arXiv: 2407.09146 [cs.LO]. URL: https://arxiv.org/abs/2407.09146.

[6] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. *The Yoneda embedding in simplicial type theory*. 2025. arXiv: 2501.13229 [cs.LO]. URL: https://arxiv.org/abs/2501.13229.

[7] Charles Rezk. "A model for the homotopy theory of homotopy theory". In: *Trans. Amer. Math. Soc.* 353.3 (2001), pp. 973–1007. DOI: 10.1090/S0002-9947-00-02653-2.

[8] Emily Riehl and Michael Shulman. "A type theory for synthetic ∞-categories". In: *Higher Structures* 1 (1 2017), pp. 147–224. DOI: 10.21136/HS.2017.06.

[9] Michael Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Mathematical Structures in Computer Science* 28.6 (2018), pp. 856–941. DOI: 10.1017/S0960129517000147. URL: https://doi.org/10.1017/S0960129517000147.

[10] Jonathan Weinberger. "A Synthetic Perspective on $(\infty, 1)$-Category Theory: Fibrational and Semantic Aspects". PhD thesis. Technische Universität Darmstadt, 2022. DOI: 10.26083/tuprints-00020716.

[11] Jonathan Weinberger. "Internal sums for synthetic fibered $(\infty, 1)$-categories". In: *Journal of Pure and Applied Algebra* 228.9 (Sept. 2024), p. 107659. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2024.107659. URL: http://dx.doi.org/10.1016/j.jpaa.2024.107659.