Realizability Triposes from Sheaves

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Abstract

Given a topos $\mathcal{E}$ and a Lawvere-Tierney topology $\square : \Omega \to \Omega$ on it, we develop a realizability $\mathcal{E}$-tripos using the internal logic of the topos. Instantiating $\mathcal{E}$ with a category of presheaves, we recover a notion of realizability with choice sequences.

Choice sequences first appeared in *Brouwer's second act of intuitionism* [14]. Brouwer envisioned an idealised mathematician that would generate entries of an infinitely proceeding sequence $(\alpha_0, \alpha_1, \alpha_2, \dots)$. At any given moment, the mathematician would only have access to the entries generated so far, hence any deductions would necessarily rely on a finite number of entries. The first formal systems of Brouwer's intuitionism were developed by Kleene and Vesley [9] and Kreisel and Troelstra [10] in which the authors investigated the Bar Theorem, continuity principles, as well as different kinds of choice sequences.

More recently, interpretations of Brouwer's choice sequences have been leveraged to give anti-classical models of dependent type theories. In [4] the authors give a computational account of forcing. Based on this view of forcing the authors of [5] produce a model of MLTT falsifying Markov's principle. This interpretation of choice sequences is combined with term models in a series of papers [1, 2, 3, 11, 7] to explore principles such as bar induction, continuity of functions on the Baire space and different versions of Markov's principle. As pointed out by [12], there is a common thread of constructions internal to sheaf models which links all the foregoing works. In the tradition of Kripke and Beth semantics, by taking a category of sheaves over a preordered set $\mathbb{W}$ of worlds and carrying out the standard operational constructions internally to this model, we should expect to recover models akin to the above.

In what follows, we will start from this observation and attempt to connect these realizability constructions, in the form of PER models with choice sequences, to categorical realizability over categories of sheaves, rather than the category of sets. As in the abstract, we fix a topos $\mathcal{E}$ and a Lawvere-Tierney topology $\square : \Omega \to \Omega$ and proceed to define a realizability tripos over $\mathcal{E}$. We refer to [8] for an introduction to topos theory, in particular section A4.4 for a treatment of Lawvere-Tierney topologies.

Definition 1. *Given objects X and Y of $\mathcal{E}$, we define* ***partial morphisms from*** X ***to*** Y *as morphisms from X to $Y_\perp$, where $Y_\perp$ is the partial map classifier of Y [8, §A2.4]. In the internal logic, given elements x and y of $X_\perp$, we use $x\downarrow$ to mean that x is defined, $x \preccurlyeq y$ to mean that if x is defined then so is y and their values agree, and $x \simeq y$ for the conjunction of $x \preccurlyeq y$ and $y \preccurlyeq x$.*

Definition 2. *An* ***internal partial combinatory algebra*** *consists of an object A of $\mathcal{E}$, a partial morphism $- \cdot - : \mathsf{A} \times \mathsf{A} \rightharpoonup \mathsf{A}$ and elements $\mathsf{k}, \mathsf{s} : \mathsf{A}$ satisfying the internal statements:*

$$\begin{array}{ccc} \mathsf{k} \cdot a \downarrow & \mathsf{s} \cdot a \downarrow & \mathsf{s} \cdot a \cdot b \downarrow \\ a \preccurlyeq \mathsf{k} \cdot a \cdot b & a \cdot c \cdot (b \cdot c) \preccurlyeq \mathsf{s} \cdot a \cdot b \cdot c & \end{array}$$

We define partial combinatory algebras (pca) using $\preccurlyeq$ as opposed to $\simeq$. It is shown in [6] that any "weak" pca, that is using $\preccurlyeq$, is isomorphic to a "strong" pca, that is using $\simeq$, hence this is mainly an aesthetic decision. Note that weak pcas differ from ordered pcas [17, §1.8].

Whereas an ordered pca comes equipped with an ordering on the underlying object A, weak pcas simply use the ordering $\preccurlyeq$ on partial terms $\mathsf{A}_\perp$.

The usual story with partial combinatory algebras carries over to the internal setting. We can still show that a pca is functionally complete and with that we get access to most programming constructs needed for realizability such as pairings, booleans, coproducts, and whatever else the mind might dream of. See [17, §1.1] for an elaboration on pcas and how to program with them.

With an internal pca we could now define a realizability tripos akin to the usual set-based realizability triposes. In fact this is done in [16], in which the author takes a pca internal to a category of presheaves, takes its sheafification to get a pca internal to the relevant category of sheaves, and then implicitly works in the realizability topos arising out of said pca object.

Definition 3. *Given an object X we define the type of* ***realizability predicates on*** *X as the type $X \to \mathcal{P}\mathsf{A}$. We further define an ordering on realizability predicates $\varphi, \psi : X \to \mathcal{P}\mathsf{A}$ by*

$$\varphi \leq \psi :\equiv \exists e : \mathsf{A}.\forall x : X.\forall a \in \varphi(x).\Box(e \cdot a \downarrow \wedge\, e \cdot a \in \psi(x))$$

Given a realizability predicate $\varphi : X \to \mathcal{P}\mathsf{A}$, for each $x : X$ we think of the subobject $\varphi(x) \hookrightarrow \mathsf{A}$ as the programs which evidence that x satisfies φ, i.e. its *realizers*. We say that φ implies ψ, written $\varphi \leq \psi$, if there exists a program e which for every x, will take evidence that x satisfies φ and convert it to evidence that x satisfies ψ. We include the modality $\Box$ to accommodate for choice sequences. For example suppose the program e relies on the sixth entry of a choice sequence α, but so far we have only generated the first three entries. In such a case, the $\Box$ lets us generate more entries for α before requiring that $e \cdot a$ be defined. If we did not use the modality then we would not be able to generate more entries in α, effectively disallowing the use of choice sequences as realizers.

Using the internal language of $\mathcal{E}$ we can show that realizability predicates on an object X with this particular ordering form a pre-ordered set. Furthermore, we can define reindexing pre-Heyting algebra morphisms by precomposition, show these have left and right adjoints (as monotone maps) satisfying the Beck-Chevalley condition, and finally we can give an appropriate generic element giving us an $\mathcal{E}$-tripos. For an introduction to the theory of realizability triposes and constructions on these we refer the reader to [17, §2]. The definitions of the required left and right adjoints, and generic element are very similar with the ones in the usual setting.

In [15] the authors also use the internal logic of a category $\mathcal{E}$ to define categories of assemblies and show that these still give models of constructive set theories. While the authors assume less about the category $\mathcal{E}$ to define assemblies, they stick to defining assemblies over the internal version of Kleene's first algebra $\mathcal{K}_1$. So while the tools used are similar, the classes of models considered are different.

To talk about choice sequences we instantiate $\mathcal{E}$ to the category of presheaves over a poset $\mathbb{W}$. In the prototypical case where we want our pca to have a single choice sequence code δ, we may take this poset to be the set of lists of natural numbers inversely ordered by prefix. The generated values of δ at each world would then be decided by the underlying list of natural numbers. For the modality $\Box$, we take the Lawvere-Tierney topology associated with the following notion of covering: an upwards closed set $\mathcal{U} \subseteq \mathbb{W}$ covers a world $w : \mathbb{W}$ if all increasing sequences of worlds starting at w intersect with $\mathcal{U}$. The category of assemblies arising out of this tripos seems like a particularly good setting for studying the realizability of choice sequences themselves, it should contain interpretations of the natural numbers and the Baire space while being simpler to work with than the realizability topos.

In this setting, an internal pca will consists of a pca A_w for each world $w : \mathbb{W}$ with application maps $(-) \cdot_w (-)$ and transition maps $(-)|_{w \sqsubseteq v} : \mathsf{A}_v \to \mathsf{A}_w$. The application maps, which are partial functions, have to be monotone in their domain as well as natural. By choosing different pca objects we expect to be able to handle different versions of choice sequences, from fully lawless choice sequences, to lawful choice sequences and any variation sitting in between these. This contrasts the situation with a sheafified pca object [16], where sheafification inadvertently adds realizers which may not have been computable before. For example, consider the pca object given by $\mathcal{K}_1$ at every world. In particular, application of terms is independent of the world and so we have no choice sequence realizers, as their behaviour will be the same at every world. If we take its sheafification, a realizer at a world w won't be a single element $e : \mathcal{K}_1$ anymore, but will instead be a compatible family of realizers $(e_u)_{u \in \mathcal{U}}$ indexed by a cover of w. Aside from being compatible [8, Definition 2.1.2 in §C2.1], we have no restrictions on this family of realizers, so sheafification has added realizers whose behaviour can vary wildly between different worlds, despite starting with a pca object without choice sequences.

Towards the study of realizability of choice sequences, we now suggest two assemblies of interest: that of pure natural numbers and that of effectful natural numbers.

Definition 4. *The **pure natural numbers assembly**, denoted $\mathsf{N}_{\mathsf{pur}}$, consists of the constant presheaf $\Delta\mathbb{N}$ along with the realizability relation $e \vDash_w n$ if and only if the code $e : \mathsf{A}_w$ equals the Church encoding of $n : \mathbb{N}$ at w.*

Definition 5. *The **effectful natural numbers assembly**, denoted $\mathsf{N}_{\mathsf{eff}}$, consists of the sheafification of $\Delta\mathbb{N}$. As for the realizability relation, if we have $\mathcal{U}$ covering w and and a compatible family n_u of natural numbers, then $e \vDash_w (n_u)_{u \in \mathcal{U}}$, if and only if there exists a cover $\mathcal{V}$ of w and for all $v \in \mathcal{U} \cap \mathcal{V}$ and $a : \mathsf{A}_v$, $(e|_{v \sqsubseteq w}) \cdot_v a$ is defined and equals the Church encoding of n_v at v.*

At the level of underlying presheaves, the later is the sheafification of the former, so we hope to find an analogous universal property to justify it as the correct definition of effectful natural numbers. With this, we can then study choice sequences as the exponential object $\mathsf{N}_{\mathsf{eff}} \to \mathsf{N}_{\mathsf{eff}}$ in this category.

We intend on using a theorem prover to formalise the arguments in the internal logic of $\mathcal{E}$ as done in [13]. The formalisation is currently in its early stages, but we hope to progress more on it once definitions are more settled.

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