A Two-level Foundation for the Calculus of Constructions

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As a foundation for mathematics, type theory is affected by a long-dated and pervasive tension between intensional and extensional representations of mathematical entities. Intuitively, mathematicians want type theories able to consider mathematical objects independently of their particular presentations, while computer scientists are more bound to these when looking for theories with good computational properties. This dichotomy has practical consequences in the proof-assistant world, with many relying on the easier-to-implement intensional theories (such as Agda, Coq, or Lean), and fewer implementing full extensional theories supporting the reflection rule (such as Nuprl or the more recent Andromeda).

In [MS05], a way out of the intensional vs. extensional impasse has been advocated through the notion of *two-level foundation*[1]. According to this paradigm, a type-theoretic foundation for mathematics should consist of two distinct type theories, each assigned with a precise role in agreement with its nature: an extensional type theory as the actual system for the foundations of mathematics, and an intensional type theory as a functional programming language in which to implement the former through a suitable interpretation. The problem of modeling full extensional type theories using intensional ones has been notably addressed in the case of Martin-Löf's type theory by Hofmann in [Hof95] in the search of conservativity results, which led to add some specific axioms to the considered intensional theory.

Always in [MS05], the authors conceived the Minimalist Foundation, a two-level foundation which can be considered either as a predicative version of the Calculus of (Inductive) Constructions [CH88, PM15], or as a version of Martin-Löf's type theory enriched with a primitive notion of proposition. The Minimalist Foundation has then been fully formalized in [Mai09] as consisting of: an extensional level **emTT** (for *extensional minimal type theory*) which extends the extensional version of Martin-Löf's in [Mar84] in particular with a power constructor, quotient sets, and proof-irrelevance; an intensional level **mTT** (for *minimal type theory*) which extends the intensional version of Martin-Löf's in [NPS90]; and an interpretation of the former in a setoid model built on the latter. The Minimalist Foundation was introduced to serve as a common-core foundation in which to develop mathematics agnostically, since it can be interpreted in many of the most relevant set-theoretic and type-theoretic foundations for mathematics; in particular, both its levels are interpretable in Homotopy Type Theory [CM22].

Here, we extend a fragment of the Calculus of Inductive Constructions $\mathbf{CC}_{\mathsf{ML}}$ supporting the basic inductive types of Martin-Löf to a two-level foundation, by taking the impredicative version $\mathbf{emTT}^{\mathsf{imp}}$ of **emTT** as its extensional level. More in detail, the theory $\mathbf{emTT}^{\mathsf{imp}}$ is obtained by adding to **emTT** an impredicative universe of propositions quotiented by logical equivalence or, equivalently, by adding a powerset constructor; we claim that such a theory pro-

[1] This has not to be confused with the notion of *two-level type theory* 2LTT [ACKS23], introduced later for a different purpose.

vides the internal language of Penon's quasitoposes and show that the results for the Minimalist Foundation proved in [Mai09] and [MS24] can be extended to it.

Firstly, we prove that $\mathbf{emTT}^{\mathsf{imp}}$ and $\mathbf{CC}_{\mathsf{ML}}$ are mutually interpretable. The main obstacle comes from the fact that, whilst resembling closely the two-level structure of Martin-Löf's type theory, where the extensional level is obtained as an extension of the intensional one with the addition of an equality reflection rule, the presence of a universe of propositions *quotiented by logical equivalence* does not make $\mathbf{emTT}^{\mathsf{imp}}$ a direct extension of $\mathbf{CC}_{\mathsf{ML}}$. Whether $\mathbf{CC}_{\mathsf{ML}}$ can be interpreted into $\mathbf{emTT}^{\mathsf{imp}}$ is therefore not trivial. We will answer it positively using the canonical isomorphisms technique applied to a bridge theory obtained by extending $\mathbf{emTT}^{\mathsf{imp}}$ with the axiom of propositional extensionality. As a byproduct, we will also conclude that the assumption $\mathsf{propext}$ of propositional extensionality is conservative over $\mathbf{emTT}^{\mathsf{imp}}$, in the sense that any proposition expressible in $\mathbf{emTT}^{\mathsf{imp}}$ and provably true in $\mathbf{emTT}^{\mathsf{imp}} + \mathsf{propext}$ is already true in $\mathbf{emTT}^{\mathsf{imp}}$.

Secondly, we prove that $\mathbf{emTT}^{\mathsf{imp}}$ is equiconsistent with its classical version through a double-negation translation. Thanks to the possibility of doing impredicative encodings in $\mathbf{emTT}^{\mathsf{imp}}$, this result includes also the translation of inductive and coinductive predicates, and therefore, according to the results in [MS23, Sab24], of (co)inductively generated formal topologies; in the predicative setting of $\mathbf{emTT}$ this is still an open problem; likewise, it is an open problem to determine if such double-negation translation can be extended to support the (co)inductive schemes of the full Calculus of Inductive Constructions.

Finally, we show how to extend the results in [Mai05] to prove that $\mathbf{emTT}^{\mathsf{imp}}$ provides the internal language of quasitoposes. The only notable change with respect to the cited work is that logic in a quasitopos must be interpreted through *strong* monomorphisms, which only in a genuine topos coincide with all monomorphisms.

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