Rezk Completions For (Elementary) Topoi

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We continue the development of univalent category theory in univalent foundations. Recall that the Rezk completion [2] provides a universal solution to the problem of constructing a univalent category from an arbitrary category. In this work, we lift the Rezk completion from categories to elementary topoi. The results presented below are formalized in (Coq-)UniMath [6][1].

1 Introduction

Internal to univalent foundations, the "well-behaved categories" are those satisfying a certain coherence condition: *univalence*. For every category, one always has a canonical function $idtoiso_{x,y}$ from the identity type $x = y$, between two objects x and y, to the type $x \cong y$ of isomorphisms between them. A **univalent category** is a category for which $idtoiso_{x,y}$ is an equivalence of types, for every x and y. The univalence requirement for categories can be motivated by to the variety of examples which satisfy this requirement, and by to the intended semantics of categories in e.g., Voevodsky's simplicial set model of HoTT/UF.

Univalent categories are particulary well-behaved. First, notions that are classically unique up to isomorphism, such as limits, become unique up to identity when working with univalent categories. Second, isomorphisms of categories coincide with equivalences. Hence, structures are automatically invariant under equivalences.

Even though many occuring categories are in fact univalent, certain constructions on categories can fail to produce univalent categories. For example, in categorical logic, the construction of a topos from a tripos (a.k.a., a second-order hyperdoctrine), often produces a non-univalent topos [3, 4]. Nonetheless, for every category $\mathcal{C}$, we can construct a univalent category $\mathsf{RC}(\mathcal{C})$ and a weak equivalence $\eta_{\mathcal{C}}$ from $\mathcal{C}$ to $\mathsf{RC}(\mathcal{C})$ [2]. That is, there is a functor $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ which is fully faithful and essentially surjective on objects; the latter condition means that for every $y : \mathsf{RC}(\mathcal{C})$, there *merely* exists some $x : \mathcal{C}$ and an isomorphism between $F(x)$ and y. Furthermore, $(\mathsf{RC}(\mathcal{C}), \eta_{\mathcal{C}})$ is universal in the sense that every functor from $\mathcal{C}$ to a univalent category $\mathcal{E}$ can be uniquely extended to a functor of type $\mathsf{RC}(\mathcal{C}) \to \mathcal{E}$. Given a category $\mathcal{C}$, the pair $(\mathsf{RC}(\mathcal{C}), \eta_{\mathcal{C}})$ is referred to as **the Rezk completion**. Motivated by the tripos-to-topos construction, we lift the Rezk completion, from categories, to categories with additional structure; in particular, to elementary topoi.

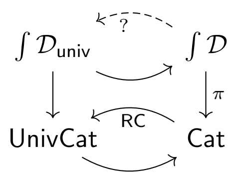
To prove that the Rezk completion lifts to topoi, it suffices that each of the ingredients defining topoi suitably transports along those weak equivalences given by the Rezk completion. This incremental proof strategy is made precise through the language of displayed (bi)categories [1].

[1] https://github.com/UniMath/UniMath/blob/master/UniMath/Bicategories/RezkCompletions/DisplayedRezkCompletions.v,https://github.com/UniMath/UniMath/tree/master/UniMath/Bicategories/RezkCompletions/StructuredCats, https://github.com/UniMath/UniMath/pull/2035

2 Framework

The universal property characterizing the Rezk completion states that every functor into a univalent category factors uniquely through the Rezk completion. The universal property implies that the inclusion of (the bicategory of) univalent categories $\mathsf{UnivCat}$, into all categories Cat, has a left biadjoint RC, whose action on objects is given by the Rezk completion. Hence, to lift the Rezk completion to categories with additional structure, we lift the left biadjoint to the bicategory whose objects are those *structured categories* and whose morphisms are *structure preserving functors*. For simplicity, we assume that the 2-cells are *all* natural transformations; an assumption shared by each of the structures characterizing elementary topoi.

Those bicategories of structured categories inherit much of the structure of the bicategory of categories. Hence, to reuse the underlying structure, we rely on the theory of displayed bicategories. To this end, let $\mathcal{D}$ be a displayed bicategory over Cat encoding some structure for categories, whose total bicategory is denoted $\int \mathcal{D}$. The restriction of $\mathcal{D}$ to $\mathsf{UnivCat}$ is denoted $\mathcal{D}_{\mathsf{univ}}$. Then, the lifting corresponds to the construction of a left biadjoint as depicted in the following diagram:



To construct the left biadjoint, we use that the unit η corresponds pointwise with weak equivalences into univalent categories. The precise properties that are to be verified are summarized in the following lemma, where local contractibility of $\mathcal{D}$ means that every type of displayed 2-cells is contractible.

Lemma 1. Let $\mathcal{D}$ be a locally contractible displayed bicategory over Cat such that

1. for every weak equivalence $G : \mathcal{C}_1 \to \mathcal{C}_2$, whose codomain is univalent, and $x : \mathcal{D}(\mathcal{C}_1)$, there is a given $\hat{x} : \mathcal{D}(\mathcal{C}_2)$ and a displayed morphism $x \to_G \hat{x}$;
2. for every univalent category $\mathcal{C}_3$, natural isomorphism $\alpha : (G \cdot H) \Rightarrow F$, terms $x_i : \mathcal{D}(\mathcal{C}_i)$ $(i = 1, 2, 3)$, and $f : x_1 \to_F x_2$, there is a given $\hat{f} : x_2 \to_H x_3$.

Then, the pseudofunctor $\mathsf{RC} : \mathsf{Cat} \to \mathsf{UnivCat}$ lifts to a biadjoint for $\int \mathcal{D}_{\mathsf{univ}} \hookrightarrow \int \mathcal{D}$.

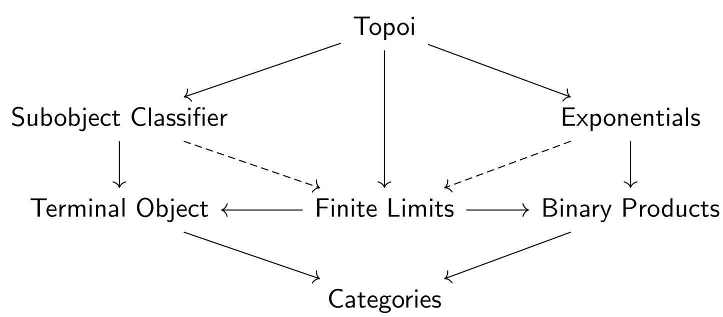
First, observe that we do not use any particular construction of the Rezk completion. The first condition in Lemma 1 states that every structure transports to the codomain of such a weak equivalence and that the functor preserves the structure. In particular, the first condition also holds for $\mathcal{C}_2 := \mathsf{RC}(\mathcal{C}_1)$ and $G := \eta_{\mathcal{C}_1}$. Hence, the Rezk completion inherits the structure given by $\mathcal{D}$ and $\eta_{\mathcal{C}_1}$ preserves the inherited structure. The second condition expresses that $(\mathsf{RC}(\mathcal{C}_1), \eta_{\mathcal{C}_1})$ not only lives in $\int \mathcal{D}_{\mathsf{univ}}$, but is universal w.r.t., those univalent categories which have the structure. Furthermore, the functoriality of $\int \mathcal{D} \to \int \mathcal{D}_{\mathsf{univ}}$ follows from the second condition. Observe that it suffices to only provide the existence part of the universality condition since the uniqueness follows necessarily from the local contractibility condition.

The assumptions of Lemma 1 are illustrated in the following example:

Example 2. Let $\mathcal{D}$ be the Cat-displayed bicategory whose displayed objects are binary products, and whose displayed morphisms witnesses the preservation of those binary products. Then, condition 1 instantiates to the following statement: For G as above, and x a choice of binary products on $\mathcal{C}_1$, $\hat{x}$ are binary products on $\mathcal{C}_2$ and G preserves those products. Condition 2 instantiates to: Given such α as above, such that F preserves binary products, then H preserves binary products.

3 Rezk Completions of Topoi

In this section, we apply Lemma 1 to lift the Rezk completion to topoi. Recall that an elementary topos is a finitely complete cartesian closed category equipped with a subobject classifier. A morphism of elementary topoi is a functor which preserves each of the structures involved. We denote the bicategory of elementary topoi as ElTop, and the bicategory of topoi whose underlying category is univalent is denoted $\mathsf{ElTop}_{\mathsf{univ}}$. In the formalization, these bicategories are constructed by stacking different displayed bicategories, starting with Cat, as depicted in the following diagram:



The main result establishes that topoi admit Rezk completions:

Theorem 3. *The inclusion* $\mathsf{ElTop}_{\mathsf{univ}} \to \mathsf{ElTop}$ *admits a left biadjoint.*

To prove Theorem 3, we apply Lemma 1 for each of the intermediary displayed bicategories. In particular, we construct Rezk completions for categories equipped with the structures mentioned above. The main challenge in constructing the left biadjoints is transporting the structure on a category along a weak equivalence, which is the first condition in Lemma 1. In particular, the univalence of categories ensures that we can apply the essential surjectiveness of the weak equivalence. The proof strategy for each of these structures follows the same steps as in Example 2.

We expect our method to apply to other kinds of structure. First, a topos possesses numerous structures, such as regularity and exactness, and local cartesian closedness, etcera. Second, there are structures only shared by some topoi, such as the existence of a natural numbers object. Third, there are other "structured categories" which are not topoi, but for which the same strategy can also be applied, e.g., abelian categories.

However, Lemma 1 cannot be applied to construct Rezk completions for structures like monoidal categories [7] and enriched categories [5]. Indeed, the contractibility assumption on the 2-cells is not generally satisfied in these cases since the natural transformations need to commute with the additional structure. Nonetheless, one could try to prove a more general lemma that does apply.

A theoretical obstacle however, is that our framework operates under a crucial assumption: that the "correct notion of univalence" for the structure in question coincides with the univalence of the underlying category. This assumption breaks down for both dagger categories, and enriched categories whose base of enrichment are not assumed to be univalent.

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