Accessible Sets in Martin-Löf Type Theory with Function Extensionality

Yuta Takahashi

Aomori University, Aomori, Japan
y.takahashi@aomori-u.ac.jp

Background: Accessible Sets in Cumulative Hierarchy After Bishop [5] showed that analysis can be developed constructively without using Brouwer's intuitionistic mathematics, several authors investigated a system to formalise Bishop's constructive analysis. For instance, Myhill [13] introduced a system of constructive set theory, and later Aczel [1, 2, 3] introduced another system of constructive set theory called constructive Zermelo-Fraenkel set theory **CZF**. On the other hand, Martin-Löf [12] took an approach different from the set-theoretic one: he formulated a framework of constructive type theory called Martin-Löf type theory **MLTT**. This framework follows the Curry-Howard correspondence, and at the same time, comprises a set of rules to define mathematical objects inductively or recursively.

Aczel's work on **CZF** also showed that these two approaches are compatible. He defined a cumulative hierarchy $\mathbb{V}$ of sets as a W-type in **MLTT**, and interpreted all axioms of **CZF** in **MLTT**. This hierarchy can be considered as a setoid model of **CZF**: $\mathbb{V}$ is a type with the equivalence relation $\doteq$, which is defined by the induction principle on $\mathbb{V}$ as a W-type. For any set $a : \mathbb{V}$, one can also define the type $\mathsf{index}\, a$ and the set $\mathsf{pred}\, a\, x$, which are the type of indices for the elements of a and the element of a of index x, respectively. We abbreviate $(i : \mathsf{index}\, a) \to \Phi(\mathsf{pred}\, a\, i)$ as $\forall_{(x \in a)} \Phi(x)$.

As in classical Zermelo-Fraenkel set theory, the transitive closure of a set can be defined in **CZF** (see, *e.g.*, [9]). Recall that the transitive closure $\mathbf{TC}(a)$ of a set a satisfies the equation

$$\mathbf{TC}(a) = a \cup \bigcup \{\mathbf{TC}(x) \mid x \in a\},$$

which implies that $\mathbf{TC}(a)$ is a transitive set:

$$\forall x \forall y (y \in x \in \mathbf{TC}(a) \to y \in \mathbf{TC}(a)).$$

So the transitive closure of a set a contains all sets below a in the hierarchy as its elements. Through Aczel's interpretation of **CZF**, one has the corresponding operator $\mathsf{tc} : \mathbb{V} \to \mathbb{V}$ in **MLTT**.

By using Dybjer's indexed inductive definition [7], one can then define the *accessibility* $\mathsf{Acc} : \mathbb{V} \to \mathsf{Set}$ with the constructor prog, the eliminator $\mathsf{ind}_{\mathsf{Acc}}$, and the computation rule below: for any universe level ℓ,

$$\begin{aligned}
\mathsf{prog} &: (a : \mathbb{V}) \to \forall_{(x \in \mathsf{tc}\, a)} \mathsf{Acc}\, x \to \mathsf{Acc}\, a \\
\mathsf{ind}_{\mathsf{Acc}} &: (P : (a : \mathbb{V}) \to \mathsf{Acc}\, a \to \mathsf{Set}\, \ell) \to \\
&\qquad ((a : \mathbb{V})(f : \forall_{(x \in \mathsf{tc}\, a)} \mathsf{Acc}\, x) \to \\
&\qquad\quad ((i : \mathsf{index}\, (\mathsf{tc}\, a)) \to P\, (\mathsf{pred}\, (\mathsf{tc}\, a)\, i)\, (f\, i)) \to P\, a\, (\mathsf{prog}\, a\, f)) \to \\
&\qquad (a : \mathbb{V})(c : \mathsf{Acc}\, a) \to P\, a\, c
\end{aligned}$$

$$\mathsf{ind}_{\mathsf{Acc}}\, P\, h\, a\, (\mathsf{prog}\, a\, f) = h\, a\, f\, (\lambda i. \mathsf{ind}_{\mathsf{Acc}}\, P\, h\, (\mathsf{pred}\, (\mathsf{tc}\, a)\, i)\, (f\, i))$$

Roughly speaking, $\mathsf{Acc}\,a$ means that the set a is constructed from below: the constructor prog says that if all sets below a are constructed then a is constructed as well. The eliminator $\mathsf{ind}_{\mathsf{Acc}}$ together with the computation rule provides the induction principle along such construction. Note that Acc is a special case of the *accessible part* of a binary relation, which can be defined as an inductive family by Dybjer's indexed inductive definition. The notion of accessible part has several applications in the areas of research such as proof theory and term rewriting (see, *e.g.*, [14] and [4], respectively).

The induction principle for Acc is stronger than the W-induction principle on $a : \mathbb{V}$ in the sense that the former admits the induction hypothesis not only for an arbitrary $x \in a$, but also for an arbitrary member of the transitive closure $\mathsf{tc}\,a$. For instance, let

$$\mathsf{nextU} : \Sigma_{(A:\mathsf{Set})}(A \to \mathsf{Set}) \to \Sigma_{(A:\mathsf{Set})}(A \to \mathsf{Set})$$

be a universe operator which takes a family of types and returns a Tarski-universe (U, T) containing all types in this family. Using the induction principle for Acc, one can define a hierarchy $\mathsf{U}\,a\,t$ of Tarski-universes with $a : \mathbb{V}$ and $t : \mathsf{Acc}$ as follows.

$$\mathsf{U}\,a\,(\mathsf{prog}\,a\,f) = \mathsf{fst}\,\Big(\mathsf{nextU}(\mathsf{index}\,(\mathsf{tc}\,a), \lambda i.\mathsf{U}\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i)\,(f\,i))\Big)$$

Roughly speaking, $\mathsf{U}\,a\,(\mathsf{prog}\,a\,f)$ contains as its subuniverses not only $\mathsf{U}\,v\,t$ for any $v \in a$ with $t : \mathsf{Acc}\,v$, but also $\mathsf{U}\,w\,s$ for any $w \in \mathsf{tc}\,a$ with $s : \mathsf{Acc}\,w$.

Though the transitive closure operator tc is accompanied by a similar induction principle which is stronger than the W-induction principle on $\mathbb{V}$, the Acc-induction principle has an advantage over that of tc: the Acc-induction principle has the simple and useful computation rule as seen above. One might try to verify that the tc-induction principle would have the following computation rule

$$\mathsf{ind}_{\mathsf{tc}}\,P\,h\,a = h\,a\,(\lambda i.\mathsf{ind}_{\mathsf{tc}}\,P\,h\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i))$$

for any $a : \mathbb{V}$ with $P : \mathbb{V} \to \mathsf{Set}\,\ell$ and $h : (a : \mathbb{V}) \to \forall_{(x \in \mathsf{tc}\,a)} P\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,x) \to P\,a$. However, in intensional **MLTT** it is implausible that one can obtain this computation rule, and in fact one should not obtain, otherwise the evaluation of $\mathsf{ind}_{\mathsf{tc}}\,P\,h\,a$ by the ξ-rule does not terminate:

$$\begin{aligned}\mathsf{ind}_{\mathsf{tc}}\,P\,h\,a &= h\,a\,(\lambda i.\mathsf{ind}_{\mathsf{tc}}\,P\,h\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i))\\ &= h\,a\,(\lambda i.h\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i)(\lambda j.\mathsf{ind}_{\mathsf{tc}}\,P\,h\,(\mathsf{pred}\,(\mathsf{tc}\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i))\,j))) = \cdots\end{aligned}$$

Aim: Accessible Sets under Function Extensionality We show that the constructor and eliminator of Acc, namely, the introduction and elimination rules of Acc are derivable in **MLTT** with function extensionality: for any universe levels ℓ_1 and ℓ_2,

$$\begin{aligned}&\mathsf{funext} : (A : \mathsf{Set}\,\ell_1)(B : A \to \mathsf{Set}\,\ell_2)\\ &\qquad (f\,g : (x : A) \to B\,x) \to ((x : A) \to f\,x =_{B\,x} g\,x) \to f =_{(x:A)\to B\,x} g.\end{aligned}$$

This means that indexed inductive definition is dispensable for formulating the constructor and eliminator of Acc in **MLTT** with function extensionality, though its computation rule is still missing without indexed inductive definition.

For this purpose, we first derive the induction principle on transitive closure tc without using function extensionality: for any universe level ℓ,

$$\mathsf{ind}_{\mathsf{tc}} : (P : \mathbb{V} \to \mathsf{Set}\,\ell) \to ((a : \mathbb{V}) \to \forall_{(x \in \mathsf{tc}\,a)} P\,x \to P\,a) \to (a : \mathbb{V}) \to P\,a.$$

The accessibility $\mathsf{Acc} : \mathbb{V} \to \mathsf{Set}$ is then defined as follows: putting $P := \lambda a.\mathsf{Set}$, we define

$$\mathsf{acc} := \lambda a.\lambda g.(i : \mathsf{index}\,(\mathsf{tc}\,a)) \to g\,i, \qquad \mathsf{Acc} := \mathsf{ind}_{\mathsf{tc}}\,P\,\mathsf{acc}.$$

Next, using function extensionality, we prove the following *propositional* computation rule of $\mathsf{ind}_{\mathsf{tc}}$. Essentially, this rule was already proved in the present author's preprint [15, Appendix].

Proposition. *For any* $P : \mathbb{V} \to \mathsf{Set}\,\ell$, $h : (a' : \mathbb{V}) \to \forall_{(x \in \mathsf{tc}\,a')} P\,x \to P\,a'$ *and* $a : \mathbb{V}$, *we have the following term of the identity type* $\mathsf{ind}_{\mathsf{tc}}\,P\,h\,a =_{P\,a} h\,a\,(\lambda i.\mathsf{ind}_{\mathsf{tc}}\,P\,h\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i))$.

$$\mathsf{comp}_{\mathsf{tc}}\,P\,h\,a : \mathsf{ind}_{\mathsf{tc}}\,P\,h\,a =_{P\,a} h\,a\,(\lambda i.\mathsf{ind}_{\mathsf{tc}}\,P\,h\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i)).$$

If one takes P and acc as above, then an instance

$$\mathsf{comp}_{\mathsf{tc}}\,P\,\mathsf{acc}\,a : \mathsf{Acc}\,a =_{\mathsf{Set}} \forall_{(x \in \mathsf{tc}\,a)} \mathsf{Acc}\,x$$

is obtained, which is crucial to our argument. Since the direction from the right of this identity type to the left corresponds to the introduction rule of Acc, we can define the constructor $\mathsf{prog} : (a : \mathbb{V}) \to \forall_{(x \in \mathsf{tc}\,a)} \mathsf{Acc}\,x \to \mathsf{Acc}\,a$ by transporting along this direction:

$$\mathsf{prog} := \lambda a.\lambda f.\mathsf{transport}\,(\lambda A.A)\,(\mathsf{sym}\,(\mathsf{comp}_{\mathsf{tc}}\,P\,\mathsf{acc}\,a))\,f.$$

To derive the Acc-elimination rule, we prove $(a : \mathbb{V})(c : \mathsf{Acc}\,a) \to P\,a\,c$ under the assumptions of this rule by induction on transitive closure of a. Transporting from $\mathsf{Acc}\,a'$ to $\forall_{(x \in \mathsf{tc}\,a')} \mathsf{Acc}\,x$ for any $a' : \mathbb{V}$ provides the function $\mathsf{inv} : (a' : \mathbb{V}) \to \mathsf{Acc}\,a' \to \forall_{(x \in \mathsf{tc}\,a')} \mathsf{Acc}\,x$ as

$$\mathsf{inv} := \lambda a'.\lambda c'.\mathsf{transport}\,(\lambda A.A)\,(\mathsf{comp}_{\mathsf{tc}}\,P\,\mathsf{acc}\,a')\,c',$$

so we have $\mathsf{inv}\,a\,c : \forall_{(x \in \mathsf{tc}\,a)} \mathsf{Acc}\,x$. By using the assumption

$$(a : \mathbb{V})(f : \forall_{(x \in \mathsf{tc}\,a)} \mathsf{Acc}\,x) \to ((i : \mathsf{index}\,(\mathsf{tc}\,a)) \to P\,(\mathsf{pred}\,(\mathsf{tc}\,a)\,i)\,(f\,i)) \to P\,a\,(\mathsf{prog}\,a\,f)$$

with the induction hypothesis $\forall_{(x \in \mathsf{tc}\,a)}(d : \mathsf{Acc}\,x) \to P\,x\,d$, we have $P\,a\,(\mathsf{prog}\,a\,(\mathsf{inv}\,a\,c))$. As a general fact on $\mathsf{transport}$, we also have

$$\begin{aligned}(A : \mathsf{Set}\,\ell_1)(P : A \to \mathsf{Set}\,\ell_2)(x\,y : A)(p : x =_A y)(c : P\,x)\\ \to \mathsf{transport}\,P\,(\mathsf{sym}\,p)\,(\mathsf{transport}\,P\,p\,c) =_{P\,x} c,\end{aligned}$$

so $\mathsf{prog}\,a\,(\mathsf{inv}\,a\,c) =_{\mathsf{Acc}\,a} c$ holds. Hence it follows from $P\,a\,(\mathsf{prog}\,a\,(\mathsf{inv}\,a\,c))$ that $P\,a\,c$ holds.

In addition, we show with function extensionality that for any $a : \mathbb{V}$ there is a unique proof of $\mathsf{Acc}\,a$, that is,

$$(a : \mathbb{V}) \to \Sigma_{(t : \mathsf{Acc}\,a)}((s : \mathsf{Acc}\,a) \to t =_{\mathsf{Acc}\,a} s)$$

holds. We formalised our result in Agda by postulating function extensionality [16].

Future Work Aczel's interpretation of **CZF** in **MLTT** was refined in Homotopy type theory (HoTT) [17]. The cumulative hierarchy $\mathbb{V}$ of sets is defined not as a W-type but as a higher inductive type, where the equivalence relation $\doteq$ on $\mathbb{V}$ is replaced with the identity type $=_{\mathbb{V}}$ and $\mathbb{V}$ has the path constructor for $=_{\mathbb{V}}$. Other interpretations of **CZF** in HoTT were investigated in, *e.g.*, [10, 11, 8]. In the literature of HoTT the accessibility in general, namely, the accessible part of a binary relation is defined by indexed inductive definition [17, 6]. A future research direction is to examine whether its special case Acc is derivable in HoTT by adapting our argument above: we will examine whether the constructor and eliminator of Acc are derivable under some interpretation of **CZF** in HoTT, which can prove function extensionality by means of the univalence axiom.

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