About the construction of simplicial and cubical sets in indexed form: the case of degeneracies

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Abstract

We presented in [HR25] a parametricity-based construction of augmented semi-simplicial and semi-cubical sets in indexed form. Since then, we refined the construction in two directions: the addition of degeneracies and a fine-grained analysis of dependencies allowing to replace some propositional equalities by definitional ones.

The now classical problem of defining semi-simplicial types in type theory[1] popularised the idea that semi-simplicial sets could alternatively be built in an "indexed way" as the following coinductive family of family of sets (here for the augmented case):

$$\begin{array}{lcl} X_{-1} & : & \mathsf{HSet} \\ X_0 & : & X_{-1} \to \mathsf{HSet} \\ X_1 & : & \Pi x : X_{-1}.\, X_0(x) \times X_0(x) \to \mathsf{HSet} \\ X_2 & : & \Pi x : X_{-1}.\, \Pi yzw : X_0(x).\, X_1(x)(y,z) \times X_1(x)(y,w) \times X_1(x)(z,w) \to \mathsf{HSet} \\ & \cdots & \end{array}$$

This amounts to see a presheaf on a direct category as a sequence of sets fibred one over the others and to iteratively apply the fibred-indexed correspondence[2]:

$$\text{(fibred)} \qquad (\Sigma T : \mathsf{HSet}.(T \to S)) \;\simeq\; (S \to \mathsf{HSet}) \qquad \text{(indexed)}$$

Furthermore, the category of augmented semi-simplices and the category of cubes, and thus augmented semi-simplicial sets and cubical sets as well, can uniformly be described as categories over ω whose morphisms between n and p are the words of length p on an alphabet $L \uplus \{0\}$ containing n times the letter 0, where L is the one-letter set $\{+\}$ in the augmented semi-simplex case and the two-letter set $\{+,-\}$ in the semi-cube case. Such description is similar to iterating Reynolds parametricity [Rey72], resulting in [HR25, Tables 1, 2, 3, 4, 5] to a uniform definition of augmented semi-simplicial and semi-cubical sets, that is, in the first case, of a family X_{-1}, X_0, X_1, ... as above. It relies on reformulating the type of each X_n under the form $\mathsf{frame}^n(X_{-1},...,X_{n-1}) \to \mathsf{HSet}$ where frame^n, defined recursively, decomposes the border of a simplex/cube into n layers made of appropriate filled simplices/cubes (we will write $\mathsf{frame}^{n,p}$ for the prefix of frame^n made of the p first layers; we will also write $\mathsf{restrframe}^{n,p}_q(d)$ for an auxiliary operation of the construction used to project along direction q a partial d : $\mathsf{frame}^{n+1,p}(X_{-1},...,X_n)$ into a $\mathsf{frame}^{n,p}(X_{-1},...,X_{n-1})$).

Adding degeneracies. The indexed construction of augmented semi-simplicial and semi-cubical sets can be equipped with degeneracies by asserting the existence of maps from X_n

[1] ncatlab.org/nlab/show/semi-simplicial+types+in+homotopy+type+theory

[2] A practical interest of such presentation of simplicial sets or cubical sets is to provide models of type theory that preserve the indexed form of dependent types, and thus liable to eventually interpret bridges or univalence definitionally.

to X_{n+1}, each applied to appropriate arguments reflecting the coherence conditions between degeneracies and faces. The degeneracies we are considering are those of usual (binary) cubical sets and of unary cubical sets as those found in parametric type theory [BCM15] (note that, along the above uniform description of augmented simplicial sets and cubical sets, degeneracies in simplicial sets are actually the unary case not of the degeneracies of cubical sets but of the connections of cubical sets!). Moreover, we consider also only one degeneracy, namely the one in the last direction of a simplicial or cubical shape (the other degeneracies could eventually be obtained by adding permutations to the structure). For a given frame d and $w : X_n(d)$, the degeneracy in the last direction $r_n(d)(w)$ has to lay on an appropriate frame for X_{n+1} whose last component is w. For instance, in the first dimensions, it takes the form:

$$\begin{array}{lcl} r_{-1} & : & \Pi x : X_{-1}.\, X_0(x) \\ r_0(x) & : & \Pi y : X_0(x).\, X_1(x)(r_{-1}(x), y) \\ r_1(x)(y,z) & : & \Pi w : X_1(x)(y,z).\, X_2(x)(r_{-1}(x), y, z)(r_0(x)(y), r_0(x)(z), w) \end{array}$$

Each r_n depends on the previous ones, so, given a sequence $(X_{-1}, X_0, ...)$ characterised in [HR25] by a coinductively-defined type $\nu\mathsf{Set}$, we coinductively define an infinitely nested Σ-type $\nu\mathsf{reflSet}$ representing the type of sequences $r_{-1}, r_0, r_1, ...$ as follows:

$$\begin{array}{l} \nu\mathsf{reflSet}(X_{-1}, X_0, ...) \quad \triangleq \\ \quad \Sigma r_{-1} : \Pi d : \mathsf{frame}^{-1}.\, \Pi x : X_{-1}(d).\, X_0(\mathsf{reflframe}^{-1}(d), x). \\ \quad \Sigma r_0 : \Pi d : \mathsf{frame}^{0}(X_{-1}).\, \Pi x : X_0(d).\, X_1(\mathsf{reflframe}^{0}(r_{-1})(d), x). \\ \quad \Sigma r_1 : \Pi d : \mathsf{frame}^{1}(X_{-1}, X_0).\, \Pi x : X_1(d).\, X_2(\mathsf{reflframe}^{1}(r_{-1}, r_0)(d), x). \\ \quad ... \end{array}$$

where

$$\mathsf{reflframe}^n(r_{-1}, ..., r_{n-1}) : \mathsf{frame}^n(X_{-1}, ..., X_{n-1}) \to \mathsf{frame}^{n+1,n}(X_{-1}, ..., X_n)$$

computes the n first layers of the border of $r_n(d)(x)$, knowing that the last layer is made of x itself, so that $(\mathsf{reflframe}^n(r_{-1}, ..., r_{n-1})(d), x)$ is a full frame, that is of type $\mathsf{frame}^{n+1}(X_{-1}, ..., X_n)$.

Note that the correct typing of $(\mathsf{reflframe}^n(r_{-1}, ..., r_{n-1})(d), x)$, relies, for $X_{-1}, X_0, ..., X_n$ being given, on two familiar coherence conditions:

$$\begin{array}{l} \mathsf{idrestrreflframe}^n(r_{-1}, ..., r_{n-1}) : \Pi d : \mathsf{frame}^n.\, \mathsf{restrframe}^{n,n}_n(\mathsf{reflframe}^n(r_{-1}, ..., r_{n-1})(d)) = d \\ \mathsf{cohrestrreflframe}^n_{p<n}(r_{-1}, ..., r_{n-1}) : \Pi d : \mathsf{frame}^{n,p}. \\ \quad \mathsf{restrframe}^{n,p}_p(\mathsf{reflframe}^{n,p}(r_{-1}, ..., r_{n-1})(d)) = \mathsf{reflframe}^{n-1,p}(r_{-1}, ..., r_{n-2})(\mathsf{restrframe}^{n-1,p}_p(d)) \end{array}$$

where $\mathsf{reflframe}^{n,p}$ generalises $\mathsf{reflframe}^n$ to prefixes of frame^n:

$$\mathsf{reflframe}^{n,p}(r_{-1}, ..., r_{n-1}) : \mathsf{frame}^{n,p}(X_{-1}, ..., X_{n-1}) \to \mathsf{frame}^{n+1,p}(X_{-1}, ..., X_n)$$

The full construction of degeneracies, machine-checked, can be found in Rocq syntax in `https://github.com/artagnon/bonak/blob/dgn/theories/` (file `νType.v`).

Getting rid of the propositional equations used in the recursive argument of [HR25]. The [HR25] construction is highly dependent. For instance, we need the definition of $\mathsf{frame}^{n,p}$ to type $\mathsf{restrframe}^{n,p}$ and we need the definition of $\mathsf{restrframe}^{n,p}$ to define $\mathsf{frame}^{n,p+1}$. In the formalisation associated to [HR25], this is managed by stating the definition of $\mathsf{frame}^{n,p}$ and $\mathsf{restrframe}^{n,p}$ propositionally in the recursive step of the construction. We are close to complete the formalisation of a propositional-equality-free variant obtained by mutually defining the *types* of all $\mathsf{restrframe}^{n,q}$ for $q < p$ together with the *definition body* of $\mathsf{frame}^{n,p}$, itself dependently on the assumption of *definitions* of $\mathsf{restrframe}^{n,q}$ for $q < p$ (and a similar mutual dependency between $\mathsf{restrframe}^{n,p}$ and the types of their coherences $\mathsf{cohrestrframe}^{n,p}$). This makes the construction much more compact.

References

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