An existential-free theory of arithmetic in all finite types

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Abstract

We introduce an existential-free theory of arithmetic in all finite types. The theory is sufficient for the soundness of the modified realizability interpretation, and hence, it can be regarded as a constructive foundational theory.

The study of arithmetic in all finite types emerged as a response to foundational problems in so-called Hilbert's program, which was an attempt to show the consistency of (formalized) mathematical theories using only "finitistic" reasoning about finite mathematical objects. By the achievement of Gödel's second incompleteness theorem, however, it was found that any consistent recursively enumerable theory which contains elementary arithmetic, does not prove its own consistency. Since finitistic reasoning (no matter what it may be) about finite mathematical objects can be formalized in elementary arithmetic, Gödel's second incompleteness theorem suggests that Hilbert's program is unattainable. Then Hilbert's program was modified to show the consistency of arithmetical theories in a manner which is as finitistic as possible. In particular, Gödel himself worked on this modified Hilbert's program. In [2] (while it is known that Gödel had the idea already in late 1930's), he introduced a quantifier-free theory of arithmetic in all finite types and reduced the consistency of first-order arithmetic PA to the consistency of a quantifier-free theory, which is known as Gödel's T. Gödel's T is a natural extension of primitive recursive arithmetic PRA which is regarded as a finitistic theory for functions over natural numbers, to functionals in all finite types, and is (more or less) acceptable as a theory which reflects finitistic standpoint in an extended sense.

On the other hand, another controversial standpoint in foundations of mathematics is constructivism. Constructivism asserts that it is necessary to construct a witness (a mathematical object) in order to prove that something exists. In the viewpoint of constructivism, a mathematical object does exist only when one can give a construction of the object. On the other hand, in the standard finitism, a mathematical object does not exist unless it can be constructed from natural numbers in a finite number of steps. Then finitism can be seen as an extreme form of constructivism. In [2], Gödel devotes a lot of space to argue that his ground of his consistency proof of arithmetic (namely, the consistency of the above mentioned T) is more plausible than Heyting's justification of his arithmetic HA, which is a counterpart of PA based on intuitionistic logic.

From a modern perspective, what Gödel established in his papers [1, 2] can be regarded as follows: In [1], he gave a translation, which is called the "negative translation" nowadays, of formal proofs of PA into formal proofs of HA. In [2], he gave a translation, which is called the "Dialectica (or functional) interpretation" nowadays, of formal proofs of HA into formal proofs of his quantifier-free theory T in all finite types. By these two steps, the consistency of PA can be reduced to the consistency of T finitistically. If one considers classical and intuitionistic arithmetic in all finite types, Gödel's achievements show the following:

1. The consistency of classical arithmetic in all finite types PA^ω is finitistically reducible to the consistency of intuitionistic arithmetic in all finite types HA^ω.

2. The consistency of HA^ω is finitistically reducible to the consistency of T.

On the other hand, in connection with constructivism, many kinds of realizability interpretation have been studied and investigated for extracting programs from proofs. In particular, Kreisel's modified (or generalized) realizability interpretation (cf. [4, 5]) is a sort of direct formalization of the notion of constructive proofs in the language of arithmetic in all finite types. In this work, we introduce an extensional theory of arithmetic in all finite types $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$, whose language is $\exists$-free (also called negative), namely, the language does not contain disjunction and existential quantifiers (of any finite types). Our theory $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ is similar to Kreisel's verification theory $\mathsf{HA}^{\omega}_{\mathsf{NF}}$ in [5] for the soundness of the modified realizability interpretation. However, our theory $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ is consistent with classical logic, while $\mathsf{HA}^{\omega}_{\mathsf{NF}}$ contains some axiom scheme on continuity which is inconsistent with classical logic, In fact, $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ is a subtheory of the extensional variant $\mathsf{E\text{-}HA}^{\omega}$ (see [3, Chapter 3]) of HA^{ω}, and T can be seen as a subtheory of $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$. In this sense, $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ is a theory in between $\mathsf{E\text{-}HA}^{\omega}$ and T. Now let $\mathsf{E\text{-}PA}^{\omega}$ be a classical extension of $\mathsf{E\text{-}HA}^{\omega}$. With respect to the Gödel-Gentzen negative translation [7, Section 1.10.2] and the modified realizability interpretation [3, Chapter 5], by induction on the length of given derivations, one can show the following:

Theorem 1. *Let A be an arbitrary $\mathsf{E\text{-}PA}^{\omega}$-formula and Δ be an arbitrary set of $\mathsf{E\text{-}PA}^{\omega}$-formulas. If $\mathsf{E\text{-}PA}^{\omega} + \Delta$ proves A, then $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}} + \Delta^N$ proves the Gödel-Gentzen negative translation A^N of A, where Δ^N is the set of the negative translations of the formulas in Δ.*

Theorem 2 (Soundness of the modified realizability interpretation)**.** *Let A be an arbitrary $\mathsf{E\text{-}HA}^{\omega}$-formula and Δ_{ef} be an arbitrary set of $\exists$-free formulas of $\mathsf{E\text{-}HA}^{\omega}$. If $\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC}^{\omega} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \Delta_{\mathrm{ef}}$ proves A, then one can extract a tuple of terms $\underline{t}$ of $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ such that $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}} + \Delta_{\mathrm{ef}}$ proves $\underline{t}$ is a modified-realizer of A and all the variables in $\underline{t}$ are contained in the free variables of A, where AC^{ω} is the scheme of choice in all finite types and $\mathrm{IP}^{\omega}_{\mathrm{ef}}$ is the independence-of-premise-schema for $\exists$-free formulas in all finite types.*

That is, the negative translation and the modified realizability interpretation (finitistically) reduce the consistency of the theories to the consistency of our theory $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$. In addition, both of the negative translation and the modified realizability interpretation for $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$-formulas (namely, $\exists$-free formulas) do not change the formulas in question anymore. Because of the facts that (i) the modified realizability interpretation is a sort of direct formalization of the notion of constructive proofs, (ii) $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ is sufficient for the verification of the soundness of the modified realizability interpretation, and (iii) the modified realizability interpretation of each theorem of $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ is the theorem itself, our existential-free theory $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$ can be regarded as a constructive foundational theory in comparison with that T is a finitistic theory (in an extended sense). From this perspective, one may argue that Gödel firstly in [1] showed the consistency of PA based on a constructive foundation, and secondly in [2], showed that based on a finitistic foundation (in an extended sense).

In addition, we introduce an axiom scheme $\mathrm{N\text{-}AC}^{\omega}_{\mathrm{ef}}$, which consists of the negative translations of the instances of AC^{ω} in the language of $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}}$. Then the soundness of the modified realizability interpretation for the negative translation of $\mathsf{E\text{-}PA}^{\omega} + \mathrm{AC}^{\omega}$ can be verified in $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}} + \mathrm{N\text{-}AC}^{\omega}_{\mathrm{ef}}$, In particular, for the interpretation of the countable choice scheme $\mathrm{AC}^{\mathbb{N}}$, only the countable fragment $\mathrm{N\text{-}AC}^{\mathbb{N}}_{\mathrm{ef}}$ of $\mathrm{N\text{-}AC}^{\omega}_{\mathrm{ef}}$ is enough, and hence, $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}} + \mathrm{N\text{-}AC}^{\mathbb{N}}_{\mathrm{ef}}$ is a constructive counterpart of T augmented with the bar recursion in the extended finitism with respect to Spector's consistency proof of classical analysis (cf. [6]). Furthermore, we show that the modified realizability interpretation of the monotone bar induction of type $\mathbb{N}$ (Kleene's formalization of Brouwer's bar theorem) can be realized in $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}} + \mathrm{N\text{-}AC}^{\mathbb{N}}_{\mathrm{ef}}$ augmented with the bar recursion for type-$\mathbb{N}$ objects. The latter is consistent with $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}} + \mathrm{N\text{-}AC}^{\mathbb{N}}_{\mathrm{ef}}$ (relative to the extended finitism with respect to Spector) but not so with $\mathsf{E\text{-}HA}^{\omega}_{\mathsf{ef}} + \mathrm{N\text{-}AC}^{\omega}_{\mathrm{ef}}$.

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