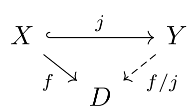
Examples and counter-examples of injective types in univalent mathematics

Tom de Jong[1] and Martín Hötzel Escardó[2]

[1] University of Nottingham, Nottingham, UK
tom.dejong@nottingham.ac.uk
[2] University of Birmingham, Birmingham, UK
m.escardo@bham.ac.uk

In the context of univalent foundations [17], injective types were introduced by the second author in [6]. The interest in injectivity originated in its use to construct infinite searchable types [5], but the topic turned out to have a rather rich theory on its own. We present new examples and counter-examples of injective types from our Agda development [7].

Definition 1. A type D is (algebraically) **injective** if it has the property that for every embedding $j : X \hookrightarrow Y$, any map $f : X \to D$ into D has a designated extension f/j. By extension, we mean that the diagram below commutes, i.e. $f/j \circ j = f$ holds.



(†)

We recall that an embedding is a map whose fibers are propositions. The algebraicity here refers to the fact that we ask for a specified extension, i.e. formulated with a Σ-type, rather than the existence of some unspecified extension, i.e. formulated with ∃, the propositional truncation of a Σ-type. Here we consider only algebraically injective types and abuse terminology by dropping the adjective.

With classical logic, i.e. in the presence of excluded middle, the injective types are precisely the pointed types. In fact, this characterization is equivalent to excluded middle. In constructive univalent foundations, the situation is more interesting, as we show in this abstract.

Injectivity and universe levels. The notion of injectivity is universe dependent [6], in the absence of propositional resizing, so that we are led, for universes $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, to consider types $D : \mathcal{W}$ that are $(\mathcal{U}, \mathcal{V})$-injective in the sense that the types X and Y in (†) are restricted to live in $\mathcal{U}$ and $\mathcal{V}$, respectively. The following theorem says that there are no non-trivial *small* injective types in general,[1] and is comparable to Corollary 10 of [1] which says that in the predicative set theory CZF it is consistent that the only injective *sets* (as opposed to classes) are singletons.

Theorem 2. If there is a $(\mathcal{U},\mathcal{U})$-injective type in $\mathcal{U}$ with two distinct points, then the type $\Omega_{\neg\neg} :\equiv \Sigma\, P : \mathcal{U}, \text{is-prop}(P) \times (\neg\neg P \to P)$ of $\neg\neg$-stable propositions in $\mathcal{U}$, whose native universe is $\mathcal{U}^+$, is equivalent to a type in $\mathcal{U}$.

For simplicity, we will not pay too much attention to the universe levels here, but we stress that they are important and that our Agda development [7] in `TypeTopology` rigorously keeps track of them.

[1] The conclusion of Theorem 2, the resizing of $\Omega_{\neg\neg}$, is not provable in univalent foundations, as observed by Andrew Swan. Given a small copy of $\Omega_{\neg\neg}$, we can interpret classical second order arithmetic via $\neg\neg$-stable propositions and subsets, but the consistency strength of univalent foundations is below that of classical second order arithmetic by [14, Corollary 6.7].

Examples. The following is a non-exhaustive list of examples of injective types and extends [6]:

1. Any *univalent* type universe $\mathcal{U}$. Indeed, given $j : X \hookrightarrow Y$ and a type family $f : X \to \mathcal{U}$, we can define $(f/j)\, y :\equiv \Sigma\,(x, -) : j^{-1}(y), f\, x$ where $j^{-1}(y) :\equiv \Sigma\, x : X, f\, x = y$ denotes the fiber of j at y. We note that defining an extension using Π instead of Σ also works.
2. The type of propositions $\Omega_{\mathcal{U}}$ in $\mathcal{U}$. Similar to above, we have extensions via Π and Σ.
3. The type of ordinals in $\mathcal{U}$, with extensions given by suprema [2, Thm. 5.8] for instance.
4. The type of iterative (multi)sets [9, 10] in $\mathcal{U}$.
5. The types of (small) ∞-magmas, monoids, and groups.
6. The type $\mathcal{L}\, X :\equiv \Sigma\, P : \Omega_{\mathcal{U}}, (P \to X)$ of partial elements [4] of any type $X : \mathcal{U}$.
7. The underlying type of any sup-complete poset, and more generally, of any pointed dcpo.

Examples 1, 2 and 6 were already present in [6], as is the fact that injective types are closed under retracts and dependent products. We now also have a sufficient criterion for a Σ-type over an injective type to be injective which accounts for the examples of Item 5. For *subtypes* of injective types there is a necessary and sufficient condition:

Theorem 3. A subtype $\Sigma\, d : D, P\, d$ of an injective type D is injective if and only if we have $f : D \to D$ such that for all $d : D$, the property P holds for $f\, d$, and $P\, d$ implies $f\, d = d$.

In particular, any reflective subuniverse [16] is injective, which also follows from [6, Theorem 24].

Counter-examples. The only type that is *provably* not injective is the empty type, because classically any pointed type is injective. But there are plenty of examples of types that cannot be shown to be injective in constructive mathematics, because their injectivity would imply a *constructive taboo*: a statement that is not constructively provable and is false in some models.

The relevant taboo in this case is **weak excluded middle** which says that for any proposition P either $\neg P$ or $\neg\neg P$ holds, and which is equivalent to De Morgan's law [11, Prop. D4.6.2].

Theorem 4. If any of the following types is injective, then weak excluded middle holds.

1. The type of booleans $\mathbf{2} :\equiv \mathbf{1} + \mathbf{1}$. (This counter-example already appears in *loc. cit.*)
2. The simple types, obtained from $\mathbb{N}$ by iterating function types.
3. The type of Dedekind reals.
4. The type of conatural [3] numbers $\mathbb{N}_\infty :\equiv \Sigma\, \alpha : \mathbb{N} \to \mathbf{2}, (\Pi\, i : \mathbb{N}, \alpha_i \geq \alpha_{i+1})$.
5. More generally, any type with an apartness relation and two points apart.

Counter-example 5 implies that none of the examples of injective types given above have interesting apartness relations. In particular, this result may be seen as an internal version of Kocsis' result [12, Corollary 5.7] that MLTT does not define any non-trivial apartness relation on a universe (in *loc. cit.* this fact is obtained by a parametricity argument).

In computation, it is important to identify decidable properties of types. The following Rice-like [15] theorem says that injective types have no non-trivial decidable properties.

Theorem 5. If an injective type has a decomposition, then weak excluded middle holds.

Here, a *decomposition* of a type X is defined to be a function $f : X \to \mathbf{2}$ such that we have $x_0 : X$ and $x_1 : X$ with $f\, x_0 = 0$ and $f\, x_1 = 1$.

While the type $\Sigma\, X : \mathcal{U}, X$ of pointed types and the type $\Sigma\, X : \mathcal{U}, \neg\neg X$ of non-empty types are both injective, the type of inhabited types is not in general.

Proposition 6. The type $\Sigma\, X : \mathcal{U}, \|X\|$ of inhabited types is injective if and only if *all propositions are projective* [13] (a weak choice principle that fails in some toposes [8]).

The above illustrates the constructive difference between the double negation and the propositional truncation (which coincide if and only if excluded middle holds).

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