Irregular models of type theory

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Propositional truncation is one of the most basic examples of higher inductive types, and formalises the idea of a proposition "most closely approximating a type" [Uni13]. We can define the propositional truncation of a type A as follows. It is a type $\|A\|$ together with a map $|-|: A \to \|A\|$, such that $\|A\|$ is a proposition and satisfies the following recursion principle: given any proposition P and map $f : A \to P$, there is a (unique) map $t : \|A\| \to P$, as illustrated below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "|-|"'] & P \\ \|A\| \arrow[ur, dotted, "t"'] & \end{tikzcd}$$

We will give two general techniques for constructing models of type theory that do *not* have propositional truncation, while retaining as much of the other parts of type theory as possible. Both techniques are based on higher modalities [RSS20], and in particular to get concrete examples of models, we will use higher modalities on cubical assemblies [Uem19, SU21] arising from Lifschitz realizability [vO08, LvO13, RS20, Kou19]. To explain the key property of Lifschitz realizability that we will use, we first recall that for a fixed universe $\mathcal{U}$, we can think of a modality as a Σ-*closed reflective subuniverse* of $\mathcal{U}$, denoted $\mathcal{U}_\bigcirc \hookrightarrow \mathcal{U}$ [RSS20, Section 1.3]. We recall [Qui16] that lex Σ-closed reflective subuniverses can be viewed as models of homotopy type theory. We note however, that the implementation of higher inductive types, and in particular propositional truncation, may or may not be preserved by the inclusion $\mathcal{U}_\bigcirc \hookrightarrow \mathcal{U}$. In particular for modalities arising from Lifschitz realizability, we can show the following theorem.

Theorem 1. *Let $\bigcirc$ be the Lifschitz realizability modality on cubical assemblies. Then,*

1. *The inclusion $\mathcal{U}_\bigcirc \hookrightarrow \mathcal{U}$ preserves Π and Σ types.*
2. *The inclusion $\mathcal{U}_\bigcirc \hookrightarrow \mathcal{U}$ preserves the empty type, coproducts and all W-types.*
3. *The inclusion $\mathcal{U}_\bigcirc \hookrightarrow \mathcal{U}$ does* not *preserve propositional truncation.*

Moreover, following [LvO13, RS20] we can construct a countable increasing sequence of reflective subuniverses $\bigcirc_n$ for $n \geq 2$ as variations on Lifschitz realizability. One can then show the following theorem.

Theorem 2. *For each $n \geq 2$,*

The inclusion $\mathcal{U}_{\bigcirc_n} \hookrightarrow \mathcal{U}_{\bigcirc_{n+1}}$ preserves Π and Σ types.

The inclusion $\mathcal{U}_{\bigcirc_n} \hookrightarrow \mathcal{U}_{\bigcirc_{n+1}}$ preserves the empty type, coproducts and all W-types.

The inclusion $\mathcal{U}_{\bigcirc_n} \hookrightarrow \mathcal{U}_{\bigcirc_{n+1}}$ does not *preserve propositional truncation.*

This already suffices to construct our simplest model of type theory without propositional truncation. We recall that when viewing locally cartesian closed categories as models of extensional type theory, the existence of propositional truncation in the type theory corresponds precisely to the locally cartesian closed category being regular [AB04, Mai05]. By simply taking the colimit over the increasing sequence of reflective subuniverses in theorem 2, one can obtain the following.

Theorem 3. *There is a category $\mathcal{E}$ such that*

1. *$\mathcal{E}$ is locally cartesian closed.*
2. *$\mathcal{E}$ has an initial object, disjoint coproducts and all W-types (including in particular a natural number object).*
3. *$\mathcal{E}$ is not regular.*

The simpler construction is limited in that it does not feature any universes of small types. Adding universes introduces multiple complications. Firstly, instead of producing models of extensional type theory, we might alternatively ask for the universe to be univalent, to get models of univalent type theory. Secondly, we might ask to have not just one universe, but a cumulative increasing sequence of universes. Finally, working with universes introduces the following subtle point about the definition of propositional truncation. For a given type $A : \mathcal{U}_n$, it might happen that A has a *restricted* propositional truncation in the sense that it does not satisfy the recursion principle in general (i.e. for all universe levels) but does once we restrict to propositions at universe level n. That is, for any proposition $P : \mathcal{U}_n$, and map $A \to P$, there is a map $\|A\| \to P$. To get models avoiding even this weaker version of propositional truncation, we use our second, more sophisticated construction, which combines the previous ideas with gluing [Shu14].

In particular, we have the following key lemma:

Lemma 4. *Suppose that $\mathcal{E}$ is a type theoretic fibration category in the sense of [Shu14] and that it moreover contains an inclusion of subuniverses $\mathcal{U} \hookrightarrow \mathcal{V}$ which preserves all type constructors except for propositional truncation, but does not preserve propositional truncation. Then viewing $\mathcal{E}^{\to}$ as a type theoretic fibration category, again as in [Shu14], it contains a universe closed under all type constructors except propositional truncation, and does not satisfy restricted propositional truncation.*

Using this and the results above on Lifschitz realizability, we can obtain the following theorems:

Theorem 5. *There is a model of type theory with a cumulative increasing sequence of universes such that:*

1. *Each universe is univalent.*
2. *Each universe is closed under Π and Σ types, coproducts and W-types, and contains the empty type.*
3. *None of the universes have all restricted propositional truncations.*

Theorem 6. *There is a model of extensional type theory with a cumulative increasing sequence of universes such that:*

1. *Each universe is closed under Π and Σ types, coproducts and W-types, and contains the empty type.*
2. *None of the universes have all restricted propositional truncations.*

As well as talking about these results, I will report on the current work in progress of adding certain higher inductive types, such as the circle type to the model.

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