Arrow algebras

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Background. An *elementary topos* can be seen as a model of a version of the Calculus of Constructions with an impredicative universe of propositions, where any two elements of a proposition are definitionally equal. There is an extensive literature on topos theory (see, for example, [10, 8, 9]) and many properties of this type theory can be proved using its topos-theoretic semantics.

An important class of such toposes is the one obtained from *locales*. A locale L is a complete poset in which the following distributive law holds:

$$a \wedge \bigvee_{b \in B} b = \bigvee_{b \in B} a \wedge b,$$

when $a \in L$ and $B \subseteq L$. Every topological space gives rise to a locale by considering its poset of open subsets ordered by inclusion.

Whenever you have a locale, you can obtain a topos from it by taking the category of *sheaves* over the locale: the result is called a *localic topos*. This category of sheaves over the locale L is equivalent to a category that has a description in terms of logic. Indeed, there is an equivalent category of L-sets, which are sets with an L-valued equality relation on them, where this equality relation is required to be symmetric and transitive; the morphisms of L-sets are L-valued functional relations.

The latter category can be understood as the result of a two-step process. First, one builds a *tripos* out of the locale L and then one turns this tripos into a topos by the *tripos-to-topos construction* [6]. Importantly, there are triposes that do not arise from locales, for instance, the effective tripos, whose associated elementary topos is Hyland's effective topos, a non-localic (even non-Grothendieck) topos [5]. The effective topos and their subcategories are important as models of polymorphic type theories [4, 7].

Contribution. The aim of this talk is to introduce *arrow algebras* and explain the work of my former MSc students Marcus Briët and Umberto Tarantino [1, 15]. Arrow algebras are algebraic structures generalising locales. The point is that they still allow you to construct a tripos, an *arrow tripos*, and hence also an *arrow topos* by the tripos-to-topos construction.

These arrow toposes include the localic toposes, but also Hyland's effective topos. Indeed, many realizability toposes can be shown to be arrow toposes, because every *pca* (*partial combinatory algebra*) gives rise to an arrow algebra: this includes also "relative, ordered" pcas as in, for example, Zoethout's PhD thesis [16] (see also [3, 13]).

Crucially, Umberto Tarantino has developed a notion of morphism of arrow algebras which correspond to geometric morphisms between the associated triposes. This has allowed us to understand the following in purely arrow algebraic terms:

1. Every arrow morphism factors as a surjection followed by an inclusion, inducing the corresponding factorisation on the level of triposes and toposes.
2. Every subtripos of an arrow tripos coming from an arrow algebra L is induced by a *nucleus* on L. Given this nucleus, there is a simple construction of a new arrow algebra inducing the subtripos.

As a result, arrow algebras provide a flexible framework for constructing and studing new toposes.

Related work. Arrow algebras can be defined as follows:

Definition 0.1. *An* arrow algebra *A is a complete lattice $(A, \preccurlyeq)$ with an implication operator $\to : A^{\mathrm{op}} \times A \to A$ and a* separator *$S \subseteq A$ such that:*

1. *if $a \in S$ and $a \preccurlyeq b$, then $b \in S$.*
2. *if $a, a \to b \in S$, then also $b \in S$.*
3. *S contains the following* combinators*:*

$$\begin{aligned}
\mathbf{k} &:= \bigcurlywedge_{a,b} a \to b \to a \\
\mathbf{s} &:= \bigcurlywedge_{a,b,c} (a \to b \to c) \to (a \to b) \to (a \to c) \\
\mathbf{a} &:= \bigcurlywedge_{a,(b_i)_{i\in I},(c_i)_{i\in I}} \big(\bigcurlywedge_{i\in I} a \to b_i \to c_i\big) \to a \to \big(\bigcurlywedge_{i\in I} b_i \to c_i\big)
\end{aligned}$$

Arrow algebras were directly inspired by Alexandre Miquel's work on *implicative algebras* [12]. His implicative algebras can be defined as arrow algebras in which the following axiom holds:

$$a \to \bigcurlywedge_{b\in B} b = \bigcurlywedge_{b\in B} a \to b.$$

We felt it was worthwhile to drop this axiom, because there are many natural examples of arrow algebras that do not satisfy it: this includes arrow algebras obtained from pcas and the arrow algebras obtained from nuclei. While it follows from Miquel's work that every arrow algebra is equivalent to an implicative algebra [1, 11], the equivalent implicative algebra is rather unwieldy and for doing concrete calculations, working with the original arrow algebra is easier.

Implicative algebras are closely related to *evidenced frames*, as in [2]. Another framework which subsumes both realizability and localic toposes is the work by Pieter Hofstra on *BCOs* (*basic combinatory objects*) [3] (see also [14]). While every implicative algebra is a BCO, it is not clear how to obtain a BCO from an arrow algebra in such a way that the associated triposes are isomorphic. However, a more thorough investigation of these connections is left to future work.

References

[1] M. Briët and B. van den Berg. "Arrow algebras". arXiv:2308.14096. 2023.

[2] L. Cohen, E. Miquey, and R. Tate. "Evidenced Frames: A Unifying Framework Broadening Realizability Models". *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021.* IEEE, 2021, pp. 1–13.

[3] P.J.W. Hofstra. "All realizability is relative". *Math. Proc. Cambridge Philos. Soc.* 141.2 (2006), pp. 239–264.

[4] J.M.E. Hyland. "A small complete category". *Ann. Pure Appl. Logic* 40.2 (1988), pp. 135–165.

[5] J.M.E. Hyland. "The effective topos". *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981).* Vol. 110. Stud. Logic Foundations Math. Amsterdam: North-Holland Publishing Co., 1982, pp. 165–216.

[6] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. "Tripos theory". *Math. Proc. Cambridge Philos. Soc.* 88.2 (1980), pp. 205–231.

[7] J.M.E. Hyland, E.P. Robinson, and G. Rosolini. "The discrete objects in the effective topos". *Proc. London Math. Soc. (3)* 60.1 (1990), pp. 1–36.

[8] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Volume 1.* Vol. 43. Oxf. Logic Guides. New York: Oxford University Press, 2002, pp. xxii+468+71.

[9] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Volume 2.* Vol. 44. Oxf. Logic Guides. Oxford: Oxford University Press, 2002, pp. xxii+469–1089+71.

[10] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic – A first introduction to topos theory.* Universitext. New York: Springer-Verlag, 1992, pp. xii+629.

[11] A. Miquel. "Implicative algebras II: Completeness w.r.t. Set-based triposes". arXiv:2011.09085. 2020.

[12] A. Miquel. "Implicative algebras: a new foundation for realizability and forcing". *Math. Struct. Comput. Sci.* 30.5 (2020), pp. 458–510.

[13] J. van Oosten. *Realizability: an introduction to its categorical side.* Vol. 152. Studies in Logic and the Foundations of Mathematics. Elsevier B. V., Amsterdam, 2008, pp. xvi+310.

[14] J. van Oosten and T. Zou. "Classical and relative realizability". *Theory Appl. Categ.* 31 (2016), Paper No. 22, 571–593.

[15] U. Tarantino. "A category of arrow algebras for modified realizability". *Theory and Applications of Categories* 44 (2025), pp. 132–180.

[16] J. Zoethout. "Computability models and realizability toposes". PhD thesis. Utrecht University, 2022.