A Curry-Howard correspondence for intuitionistic inquisitive logic

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Abstract

In this paper, we introduce a typed natural deduction system for propositional intuitionistic inquisitive logic. The term calculus we use to establish a Curry-Howard correspondence is lambda calculus extended with a new construct corresponding to the logical behaviour of the Split rule, a key rule of inquisitive logic. We show that the resulting system is normalizing. The existence of this system corroborates previous observations that questions have constructive content.

Extended abstract

Inquisitive logic [2] is a framework that accounts for both statements and questions within a unified formal system. In recent years, research on inquisitive logic has grown significantly and it has found its application in many other areas such as linguistics or philosophy of language [3].

Inquisitive logic is especially well-explored and understood from model-theoretic and algebraic points of view [12, 4]. Recently, there has been progress in the proof-theoretic understanding of the system [13, 7]. However, when it comes to a type-theoretic view, the picture of inquisitive logic becomes less clear. To our knowledge, this area has not yet been properly explored.

In this paper, we want to fill this gap and examine inquisitive logic from a type-theoretic point of view. The canonical system of inquisitive logic is based on classical logic of statements. However, in the type-theoretic context, it is natural to focus instead on intuitionistic inquisitive logic (InqIL), which is an inquisitive logic of questions based on intuitionistic logic of statements [5, 9, 10]. In particular, we introduce a Curry-Howard correspondence between a natural deduction system for propositional InqIL and a lambda calculus extended with a new construct select that will capture the logical behaviour of the key rule of inquisitive logic called Split.

The fact that this can be achieved shows that there is a close link between the notions of inquisitive/interrogative content and computational/constructive content. This was already partly hinted at by the following Ciardelli's observation:

> proofs [of dependencies; i.e., proofs where both premises and conclusion are questions] have an interesting kind of constructive content, reminiscent of the proofs-as-programs interpretation of intuitionistic logic: a proof of a dependency encodes a method for *computing* the dependency, i.e., for turning answers to the question premises into an answer to the question conclusion. ([2], p. 3; see also [1], p. 324)

This also raises a further intriguing question: can these approaches be combined? For example, while inquisitive semantics can model many kinds of questions, it is unsuitable for deductive or computational ones (e.g., "What is 2 + 2?" or "What follows from φ?") as its key semantic notion of informational support is closed under logical consequence, and thus computational questions are trivially resolved by all information states. A type-theoretic approach, however, has tools for tackling these issues.

In the basic version of intuitionistic inquisitive logic, inquisitive disjunction is the primary question-forming operator and there is no primitive declarative disjunction. However, declarative disjunction can be added to the system, either in the form of the so-called tensor disjunction, as in [5, 10], or it can be defined as presupposition of the inquisitive disjunction, if we add to the language the presupposition modality $\circ$, as in [11]. We will also discuss these extensions.

We obtain InqIL by extending intuitionistic propositional logic (IPL) by the following rule known as Split:

$$\frac{\alpha \to (\varphi \vee \psi)}{(\alpha \to \varphi) \vee (\alpha \to \psi)}\ \text{Split}$$

where φ and ψ are arbitrary formulas corresponding to either questions or statements and α is a declarative, that is, a disjunction-free formula corresponding to a statement. The rule intuitively says that conditional questions are disjunctive questions resolved by suitable conditionals. In particular, the possible answers to the conditional question *whether q or r, if p* are, in accordance with Split, the conditionals *if p then q* and *if p then r*.

A key step in obtaining a Curry-Howard correspondence for InqIL rests on finding an appropriate constructive function that would capture the behaviour of the key Split rule from a computational point of view. To this end, we can utilize the results of [8] that introduced a generalized version of this rule with such a function in the context of a propositional fragment of Martin-Löf's constructive type theory.

We show that we can carry over this function into InqIL and use it to derive the Split rule and thus also provide its computational interpretation. The resulting lambda term capturing its behaviour will be as follows:

$$\dfrac{\dfrac{f : \alpha \to (\varphi \vee \psi) \qquad [x : \alpha]}{\mathsf{ap}(f, x) : \varphi \vee \psi} \qquad \dfrac{[y : \alpha \to \varphi]}{\mathsf{injl}(y) : (\alpha \to \varphi) \vee (\alpha \to \psi)} \qquad \dfrac{[z : \alpha \to \psi]}{\mathsf{injr}(z) : (\alpha \to \varphi) \vee (\alpha \to \psi)}}{\mathsf{select}(x.\mathsf{ap}(f, x), y.\mathsf{injl}(y), z.\mathsf{injr}(z)) : (\alpha \to \varphi) \vee (\alpha \to \psi)}$$

The crucial new construct select is defined as follows:

$$\dfrac{\begin{matrix}[x : \alpha] \\ c(x) : \varphi \vee \psi\end{matrix} \qquad \begin{matrix}[y : \alpha \to \varphi] \\ d(y) : \chi\end{matrix} \qquad \begin{matrix}[z : \alpha \to \psi] \\ e(z) : \chi\end{matrix}}{\mathsf{select}(x.c(x), y.d(y), z.e(z)) : \chi}$$

with the following computation rules, where $t(x) : \varphi$ and $s(x) : \psi$:

$$\mathsf{select}(x.\mathsf{injl}(t(x)), y.d(y), z.e(z)) \Rightarrow d(\lambda x.t(x))$$
$$\mathsf{select}(x.\mathsf{injr}(s(x)), y.d(y), z.e(z)) \Rightarrow e(\lambda x.s(x))$$

Note that the new construct select is a variable-bining operator (the notation '$x.c(x)$' means that the variable x becomes bound in $c(x)$ by select) and that it is treated as an eliminatory noncanonical operator for disjunction (without appearances of the declarative formula α, the rule reduces to the standard disjunction elimination rule). From a functional perspective, adding select allows us to compute even open terms to canonical values, as long as the free variables of those open terms range over declarative formulas α only.

Furthermore, we consider a variant of InqIL called InqIL° extended with a presupposition modality $\circ$. This modality was introduced in [11] and defined via the following rules:

$$\frac{\varphi}{\circ\varphi}\ \circ\mathrm{I} \qquad\qquad \dfrac{\circ\varphi \qquad \begin{matrix}[\varphi]^i \\ \alpha\end{matrix}}{\alpha}\ \circ\mathrm{E}_i$$

This modality is inspired by the truncation modality from homotopy type theory [6] that turns types into mere propositions, that is, types that are inhabited by at most one term (up to equivalence). The presupposition modality $\circ$ turns inquisitive formulas into declarative ones. As mentioned above, it can be used to define declarative disjunction as $\circ(\varphi \vee \psi)$. And, analogously to Split and select, we introduce new constructs for $\circ$I and $\circ$E called pre and sup that will allow us to extend the Curry-Howard correspondence to InqIL° as well.

Having a typed natural deduction system for inquisitive logic makes it possible to utilize Tait's computability method for proving normalization [14]. Specifically, as the notion of reduction $\Rightarrow$ that we will define via computation rules is a deterministic weak head reduction (that is, there is at most one possible reduction for any given term), we show that both InqIL and InqIL° are weakly normalizing. However, we suspect that if a more general notion of reduction is adopted, the strong normalization property can be obtained as well.

Finally, establishing a Curry-Howard correspondence for inquisitive logic sheds further light on the connection between the *formulas-as-types* principle innate to the Curry-Howard correspondence and the *questions-as-information types* interpretation of inquisitive logic [1]. First, our results confirm that formulas of inquisitive logic can indeed be regarded as types, as previously suggested by Ciardelli [1], p. 352 (see also [2], p. 110).

Second, the inherent distinction between information types and singleton types of inquisitive logic can be seen as paralleling the type-theoretic distinction between types and mere propositions. From this perspective, both truncation of type theory and presupposition of inquisitive logic can be seen as operators for suppressing content: computational one in the case of truncation and inquisitive one in the case of presupposition.

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